SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
Area of Mathematics

On the desingularization of Kähler orbifolds with constant scalar curvature

Ph.D. Thesis

Supervisor
Prof. Claudio Arezzo
ICTP

Internal supervisor
Prof. Ugo Bruzzo

Candidate
Riccardo Lena

Submitted in partial fulfillment of the requirements
for the SISSA degree of "Philosophiæ Doctor"

ACADEMIC YEAR 2012/2013

Introduction

Every smooth manifold M admits a Riemannian metric, actually a lot of them, but there is not a canonical one. So a natural question is: "Which is the best metric to put on M?". The answer, clearly, depends upon the meaning we give to the word "best", indeed the "best" could be the one compatible with some additional structure on M, or the one satisfying some conditions, that could be local or global, on certain kind of curvatures. A natural request for a "best" metric is to make the manifold the most symmetric as possible and this translates in some uniformity conditions on some types of curvatures. Every metric g on M determines uniquely a preferred connection, the Levi-Civita conection ∇, and an associated $(4,0)$-tensor field: the Riemann tensor

$$R(X,Y,Z,W) := g(\nabla_X \nabla_Y Z, W) - g(\nabla_Y \nabla_X Z, W) - g\left(\nabla_{[X,Y]} Z, W\right) \qquad X, Y, Z, W \in TM\,,$$

that encodes all curvature properties of M with respect to g. The tensor R has many symmetries that make it a bilinear form on the bundle of tangent 2-planes to M, that is

$$\mathcal{R} : \Lambda^2 TM \times \Lambda^2 TM \to \mathbb{R}\,,$$

$$\mathcal{R}(X \wedge Y, Z \wedge W) := R(X,Y,Z,W)\ .$$

A first possible request for a metric is to have the same curvature at every 2-tangent plane, and this is equivalent to requiring the sectional curvature K

$$K(X \wedge Y) := \frac{\mathcal{R}(X \wedge Y, X \wedge Y)}{|X \wedge Y|_g^2} \qquad X, Y \in TM\,,$$

to be a constant. This condition imposes further symmetries on the Riemann tensor, indeed we find that

$$\mathcal{R}(X \wedge Y, X \wedge Y) \equiv K\,|X \wedge Y|_g^2\ .$$

It turns out that requiring the sectional curvature to be constant is quite restrictive, indeed the only manifolds admitting metrics of constant sectional curvature are the sphere, the hyperbolic space, the euclidean space and their quotients by discrete subgroups of their groups of isometries. In general, if we impose conditions on the sectional curvature of a manifold, we automatically impose conditions on its topology. For example: if we want the sectional curvature to be strictly positive, then the manifold has to be compact with finite fundamental group, if we want the sectional curvature to be always less than or equal to 0 then the universal cover of the manifold has to be the euclidean space. We can weaken our requests and ask for uniformity of a kind of curvature that encodes "less information" about the topology of the manifold, to this aim we can consider the scalar curvature s_g

$$s_g : M \to \mathbb{R}\,,$$

that we obtain tracing repeatedly the Riemann tensor. If we are given a manifold M is it possible to find a metric with constant scalar curvature? This is the celebrated Yamabe problem and it has been completely solved in the 80's for the case of M compact by Trudinger, Aubin and Schoen. So the answer to our question is affirmative, if M is compact then every metric is conformal to a metric with constant scalar curvature.

Another possible notion of the "best" metric is a metric that has some minimizing property, i.e. a solution of a certain optimization problem. More precisely let M be a compact manifold and let $\mathfrak{Met}(M)$ be the space of metrics on M, then we can consider functionals

$$\mathcal{F} : \mathfrak{Met}(M) \longrightarrow \mathbb{R}\,,$$

and look for their critical points that, provided they exist, we will call *critical metrics.* An important and well studied functional of this kind is

$$\mathcal{F}(g) := \frac{1}{\mathrm{Vol}_g(M)^{\frac{n-1}{n}}} \int_M s_g \, d\mu_g\,.$$

If we write Euler-Lagrange equation for $\mathcal{F}$ we find that its critical points have to satisfy the equation

$$\mathrm{Ric}_g = \frac{s_g}{n} g\,,$$

and so the critical points are the Einstein metrics. In dimension 2 and 3, requiring a metric to be Einstein is the same as requiring constant sectional curvature so $\mathcal{F}$ can't have critical points unless the manifold is a space form. We have topological obstructions to the existence of Einstein metrics also in dimension 4, indeed if a 4-manifold M admits an Einstein metric the following inequality (Hitchin-Thorpe) holds

$$\chi(M) \geq \frac{3}{2} |\tau(M)|\,,$$

where $\chi(M)$ is the Euler characteristic of M and $\tau(M)$ its signature. In dimension greater or equal than five the question is widely open, indeed there is no known obstruction for the existence of Einstein metrics.

If we consider complex manifolds we look for our "best" metric among those that are compatible with the complex structure and so among the Hermitian ones. A remarkable feature of the complex structure is that it gives a 1-1 correspondence between hermitian 2-tensors and real differential 2-forms, so we can translate every problem involving hermitian metrics into one involving differential forms. If our manifold M is compact and Kähler, and so we fix a complex structure J and a positive closed differential 2-form ω it is natural to search for "best metrics" in the the cohomology class $[\omega] \in H^2(M, \mathbb{R}) \cap H^{(1,1)}(M, \mathbb{C})$. On compact Kähler manifolds holds the $i\partial\overline{\partial}$-lemma and this tells us that our preferred metric ω', provided it exists, is of the form

$$\omega' = \omega + i\partial\overline{\partial} f \qquad f \in C^\infty(M)\,.$$

As in the real case we can look for *critical metrics*, and if $[\omega]$ is a fixed Kähler class on M and $\omega \in [\omega]$ is a positive real 2-form then the domain of our functionals will be

$$\mathfrak{M}_\omega := \left\{ f \in C^\infty(M) \,|\, \omega + i\partial\overline{\partial} f > 0 \right\}\,.$$

The first natural functional to study would be again

$$\mathcal{F}(f) := \frac{1}{n!\mathrm{Vol}_{\omega+i\partial\overline{\partial} f}(M)^{\frac{2n-1}{2n}}} \int_M s_{\omega+i\partial\overline{\partial} f} \left(\omega + i\partial\overline{\partial} f\right)^n\,,$$

but it turns out that $\mathcal{F}$ is constant on $\mathfrak{M}_\omega$ since quantities involved are cohomological, indeed

$$\mathrm{Vol}_{\omega+i\partial\overline{\partial}f}(M) = \mathrm{Vol}_\omega(M) = [\omega]^{\cup n}$$

and

$$\int_M s_{\omega+i\partial\overline{\partial}f} \frac{\left(\omega+i\partial\overline{\partial}f\right)^n}{n!} = \int_M s_\omega \frac{\omega^n}{n!} = \frac{4\pi}{(n-1)!}[c_1(M)] \cup [\omega]^{\cup n-1}.$$

Since $\mathcal{F}$ does not distinguish different metrics in $\mathfrak{M}_\omega$, Calabi in [Cal85] proposed to study the following functional on $\mathfrak{M}_\omega$

$$\mathcal{F}'(f) := \int_M s^2_{\omega+i\partial\overline{\partial}f} \frac{\left(\omega+i\partial\overline{\partial}f\right)^n}{n!}.$$

Its Euler-Lagrange equation tells us that the critical points are metrics $\omega' \in [\omega]$ that satisfy

$$\overline{\partial}\partial^\sharp s_{\omega'} = 0$$

and we call them *Extremal Kähler metrics*. A Kähler metric is extemal if and only if the $(1,0)$-part of the gradient of the scalar curvature is a holomorphic vector field and clearly this is the case if the scalar curvature is constant. In light of this we can formulate the Kähler analogue of Yamabe problem: if we are given a compact Kähler manifold (M,ω) can we find a metric $\omega' \in [\omega]$ with constant scalar curvature? Contrarily to the real case the answer is no in general, indeed the constancy of scalar curvature implies strong conditions on the analytic structure of the manifold. Matsushima and Lichnerowicz showed, indeed, that if a Kähler manifold (M,ω) has constant scalar curvature, then the identity component of the biholomorphism group of M is a reductive complex Lie group. The main source of examples of cscK manifolds (Kähler with constant scalar curvature) are Kähler-Einstein manifolds that are Kähler manifolds whose metric satisfies the identity

$$\mathrm{Ric}_g = \frac{s_g}{2n}\omega_g$$

and it is quite hard to find explicit examples of cscK manifolds that are not in this class. Arezzo and Pacard in [AP06] and [AP09] introduced a generalized connected sum construction that allows, under suitable hypotheses, to produce new cscK manifolds starting from a given one. More precisely they prove the following two theorems.

Theorem. *Let (M,g) a compact cscK orbifold with isolated singularities that is Futaki non degenrate. Let $n \geq 1$, $p_1, \dots p_n \in M$ with neighborhoods biholomorphic to neighborhoods of the origin of $\mathbb{C}^m/\Gamma_i$ with $\Gamma_i \triangleleft U(m)$ finite (even trivial). Suppose that every $\mathbb{C}^m/\Gamma_i$ admit an ALE Kähler resolution (X_i, η_i). Then there exist an $\varepsilon_0 > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon_0)$ there exist a constant scalar curvature Kähler metric g_ε on the space*

$$\tilde{M} = M \sharp_{p_1} X_1 \cdots \sharp_{p_n} X_n,$$

that as ε tends to 0 the sequence g_ε converges to g in C^∞ topology away from points p_i. If s_g is positive or negative so is s_{g_ε} Moreover if M has no holomorphic vector fields vanishing somewhere

$$[\omega_{g_\varepsilon}] = \pi^*[\omega_g] + \varepsilon^2 \left(\sum_{k=1}^n [\eta_k]\right),$$

with $\pi : \tilde{M} \to M$ the canonical holomorphic surjection.

Theorem. *Let (M, g) be a cscK manifold, and let $\varphi_1, \dots, \varphi_d \in C^\infty(M)$ such that*

$$\ker(\mathbb{L}_g) := span_{\mathbb{R}} \{1, \varphi_1, \dots, \varphi_d\} \, .$$

Let $p_1, \dots, p_n \in M$ with $n \geq d+1$ and

$$\Phi = (\varphi_i(p_j))_{1 \leq i \leq d, 1 \leq j \leq n} \, .$$

If

$$rk(\Phi) = d$$

and there exist $\boldsymbol{a} := (a_1, \dots, a_n) \in (\mathbb{R}^+)^n$ satisfying

$$\Phi \boldsymbol{a} = 0 \, ,$$

then there exist ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ there exist a cscK metric g_ε on $Bl_{p_1, \cdots, p_n} M$ such that

$$[\omega_{g_\varepsilon}] = \pi^* [\omega_g] - \varepsilon^2 \left(\sum_{k=1}^{n} \hat{a}_k^{\frac{1}{m-1}} [c_1(\mathcal{O}(E_k))] \right)$$

with E_k the exceptional divisor at point p_k and

$$\hat{a}_k \to a_k \qquad as\ \varepsilon \to 0 \, .$$

The first theorem says that if the starting cscK orbifold has no holomorphic vector fields or it is Futaki non-degenerate, then we can replace smooth points with projective spaces (blow ups) and singular points with suitable (provided they exist) ALE Kähler scalar flat resolutions. ALE spaces are manifolds "very similar" to euclidean spaces indeed, outside a compact set they are biholomorphic to a quotient by a finite subgroup of $SU(m)$ of the complement of a ball in $\mathbb{C}^m$. The strategy of proof of the first theorem is roughly the following:

- cut small balls $B_{r_\varepsilon}(p)$ centered on points we want to replace;
- choose a big compact subsets of the relative ALE manifolds;
- construct a family of cscK metrics depending on parameters on the truncated starting cscK manifold;
- construct families of cscK metrics depending on parameters on the truncated ALE spaces;
- find right parameters such that the metrics on truncated starting cscK manifold and the truncated ALE spaces glue.

The families of metrics above are constructed essentially in two steps: first modifying smartly "by hand" background metrics and then solving carefully constructed PDEs. These PDEs are solvable because the assumption that the starting cscK orbifold has no holomorphic vector fields or it is Futaki non-degenerate translates in the invertibility of a certain linear differential operator. In the second theorem the assumption on holomorphic vector fields is replaced by a geometric condition relating holomorphic vector fields with points we want to replace and their "position" in the manifold. The strategy of proof of the second theorem is morally the same of the first but it is much more delicate. Indeed the linear differetial operator we mentioned above is not easily invertible any more and a more refined argument is needed for constructing the families of metrics. The ALE structure of the blow up of euclidean spaces plays a crucial role in the proof. Indeed the analysis of asymptotics of the Burns-Simanca metric is the key fact that allows to find a right inverse for a differential operator that is necessary to the construction the families of cscK metrics. There are many related works on this subject, in [RS09a] Rollin and Singer construct cscK surfaces using the concept of parabolic polystability and state a generalization of Theorem 1.6, in [APS11] Arezzo, Pacard and Singer study the natural generalization of Theorem 1.6 to the case of extremal metrics, indeed they prove that if a Kähler manifold has an extremal metric then, under geometric assumptions similar to those of Theorem 1.6, its blow up carries an extremal Kähler metric. In [Sto10], Stoppa shows that, in the projective setting, the conditions of Theorems 1.5 and 1.6 are also necessary in order to have a cscK metric on the blown up manifold. Szekelyhidi, in [Szé12], generalizes [APS11] showing that the blowup of an extremal Kähler manifold at a relatively stable point in the sense of GIT admits an extremal metric in Kähler classes that make the exceptional divisor sufficiently small. The purpose of this thesis is to extend Theorems 1.5 and 1.6 to the case of cscK orbifolds with isolated singularities and with holomorphic vector fields. These objects can arise as Gromov-Hausdorff limits of Fano Kähler-Einstein manifolds and it is natural to ask if they can be desingularized, at least partially, in such a way they remain cscK. Our main result in this direction is the following theorem.

Theorem 1.7. *Let (M, g) be a compact cscK orbifold with isolated singularities and let*

$$\boldsymbol{p} := \{p \in M | p \text{ is a } SU(m) \text{ singularity admitting a Kähler crepant resolution}\}$$

and

$$\ker(\mathbb{L}_g) = \langle 1, \varphi_1, \dots, \varphi_d \rangle \,.$$

Suppose moreover that

- $\sharp \boldsymbol{p} = N \geq d + 1$;
- *the $d \times N$ matrix*
$$\Delta\Phi(\boldsymbol{p})_{i,j} := \Delta_g \varphi_i(p_j) + s_g \varphi_i(p_j) \qquad (1.9)$$
has full rank;
- *there exist $\boldsymbol{b} := (b_1, \dots, b_N) \in \mathbb{R}_+^N$ such that*
$$\sum_{j=1}^{N} b_j \left[\Delta_g \varphi_i(p_j) + s_g \varphi_i(p_j)\right] = 0 \quad 1 \leq i \leq d\,. \qquad (1.10)$$

Then there exist $\left(\tilde{M}, \tilde{g}_{\boldsymbol{b}}\right)$ cscK orbifold together with a holomorphic, surjective

$$\pi : \tilde{M} \to M\,.$$

The manifold $\tilde{M}$ is obtanied replacing $\boldsymbol{p}$ with ALE-Kähler spaces that are Ricci-flat.

The proof of Theorem 1.7 is quite technical and we suggest to read Subection 1.6.2, where we give a detailed explanation of the strategy we will follow, and Section 1.7 where we fix the notation we will use in this work. The main difficulty in proving this result is that we are replacing singular points and, contrarily to Theorem 1.6, the associated *ALE* spaces are essentially different from blow up of euclidean space. Indeed, these ALE spaces are not only scalar flat as blow up of $\mathbb{C}^m$ but they are indeed Ricci-flat and this implies that there is an asyptotic growth missing in the expansion “at infinity” of the metric. This missing asymptotic was the key fact in proving Theorem 1.6. This fact forces us to find a new geometric condition that assures invertibility of differential operators that we will need to construct families of metrics on truncated spaces. A new geometric condition is not enough to follow the lines of proof of Theorem 1.6 indeed we need a more careful construction of families of metrics on truncated spaces since we now need "better" estimates to be able to prove Theorem 1.7. The conditions (1.9) and (1.10) can be rephrased in terms of the moment map for the action on M of the group $\mathfrak{H}$ of Hamiltonian isometries of (M, ω_g). Indeed let $\mathfrak{h}^*$ be the dual Lie algebra of $\mathfrak{H}$, let

$$\mu : M \longrightarrow \mathbb{R}^d ,$$

be the moment map with the identification $\mathfrak{h}^* \simeq \mathbb{R}^d$, then condition (1.9) becomes

$$\mathfrak{h}^* = \mathrm{span}_{\mathbb{R}} \left\{ (\Delta_g \mu + s_g \mu)(p_1), \ldots, (\Delta_g \mu + s_g \mu)(p_N) \right\}$$

and condition (1.10) becomes

$$\sum_{j=1}^{N} b_j \left(\Delta_g \mu + s_g \mu \right)(p_j) = 0 .$$

The laplacian of the moment map appears in [Szé12] in the context of blow ups of extremal Kähler manifolds. It comes out in the "second order" expansion of the Futaki invariant of the blow up of an extremal Kähler manifold and he makes the following conjecture.

Conjecture ([Szé12])**.** *Suppose that M admits a cscK metric in $c_1(L)$, and let $p \in M$. There exist $\varepsilon_0, \delta_0 > 0$ such that if $\mu(p) + \delta\Delta\mu(p) = 0$ for some $\delta \in (0, \delta_0)$ then for all $\varepsilon \in (0, \varepsilon_0)$ the manifold Bl_pM admits a cscK metric in the Kähler class $c_1(\pi^*L - \varepsilon E)$.*

In [Szé13] he proves this conjecture, more precisely he proves the following theorem.

Theorem. *Assume that the dimension $m > 2$, and suppose that $\nabla s(\omega)$ vanishes at $p \in M$. There is a $\delta_0 > 0$ depending on (M, ω) with the following property. Suppose that for some $\delta \in (0, \delta_0)$ there is a point q in the G^c-orbit of p such that the vector field $\mu(q) + \delta\Delta\mu(q)$ vanishes at q. Then the blowup Bl_pM admits an extremal metric in the Kähler class*

$$\pi^*[\omega] - \varepsilon^2[E]$$

for all sufficiently small $\varepsilon > 0$.

So $\Delta\mu$ plays a crucial role for the existence of an extremal metric on the blow up of an extremal Kahler manifold. The presence of the laplacian of the moment map in the second order expansion of the Futaki invariant of Bl_pM is deeply related to the structure of the Burns-Simanca metric on $Bl_0\mathbb{C}^m$. We recall that the Burns-Simanca metric η_0 on $Bl_0\mathbb{C}^m$ is an ALE metric with expansion

$$\eta_0 = i\partial\overline{\partial} \left(\frac{|x|^2}{2} - |x|^{4-2m} + J|x|^{2-2m} + \mathcal{O}(|x|^{6-4m}) \right)$$

and every Kähler metric $\tilde{\omega}$ on Bl_pM on a small neighborhood of the exceptional divisor is $i\partial\overline{\partial}$-cohomologous to a positive multiple of η. The link between between expansion of Futaki invariant of Bl_pM with a Kähler metric $\tilde{\omega} \in [\pi^*\omega] - \varepsilon^2[c_1(E)]$ is given by the following heuristic correspondences

$$\frac{d}{d\varepsilon} \mathrm{Fut}\left(\cdot, [\pi^*\omega] - \varepsilon^2[c_1(E)] \right) \bigg|_{\varepsilon=0} \leftrightsquigarrow \text{ asymptotic } |x|^{4-2m} \text{ of } \eta_0 ,$$

that in [Sto09] and [Szé12] it is shown to depend only on the moment map μ and

$$\frac{d^2}{d\varepsilon^2}\mathrm{Fut}\left(\cdot,[\pi^*\omega]-\varepsilon^2[c_1(E)]\right)\Big|_{\varepsilon=0} \leftrightsquigarrow \text{ asymptotic } |x|^{2-2m} \text{ of } \eta_0\,,$$

that Szekelyhidi in [Szé12] and [Szé13] shows it depends on the moment map μ and its laplacian $\Delta\mu$. The above relations become clear when one compute the Futaki invariant by means of the localization formulæ(see [Tia00], [BGV04]). In [Szé13], Szekelyhidi deals with second order expansion of the Futaki invariant and many of his constructions are similar to ours. This is not surprising since he studies objects strictly related to the asymptotic $|x|^{2-2m}$ of η_0 that is exactly the first non trivial asymptotic of the model spaces we use to replace singular points. As a corollary of the theory we develope to prove the Theorem 1.7 we get also the following result that Rollin and Singer in [RS09a] state without proof.

Theorem 4.2. *Let (M,g) be a cscK orbifold with isolated singularities, and let $\varphi_1,\dots,\varphi_d \in C^\infty(M)$ such that*

$$\ker(\mathbb{L}_g) := span_{\mathbb{R}}\{1,\varphi_1,\dots,\varphi_d\}\,.$$

- *Let*

$$\boldsymbol{p} := \{p_1,\dots,p_n\} \subseteq M \qquad n \geq d+1\,,$$

  a set of points with negighborhoods biholomorphic to B_1/Γ_p with Γ_p a finite subgroup, even trivial, of $U(m)$. Moreover let $\mathbb{C}^m/\Gamma_p$ admit a scalar flat ALE resolution (X_p,η_p) (in the case Γ_p is trivial we consider the blow up) such that the metrics η_p have asymptotic expansion

$$\omega_{\eta_p} = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + E_p|z|^{4-2m} + J_p|z|^{2-2m} + \mathcal{O}\left(|z|^{-2m}\right)\right) \qquad E_i \neq 0\,,$$

  and for $m=2$

$$\omega_{\eta_p} = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + E_p\log(|z|) + J_p|z|^{-2} + \mathcal{O}\left(|z|^{-4}\right)\right) \qquad E_i \neq 0\,.$$

- *Let $\boldsymbol{q} \subseteq M$ the set of points with negighborhoods biholomorphic to B_1/Γ_q with Γ_q a finite subgroup of $SU(m)$ and such that $\mathbb{C}^m/\Gamma_q$ admit an ALE Kähler crepant resolution (Y_q,θ_q).*

Let

$$\Phi = \left(\frac{E_{p_j}}{|E_{p_j}|}\varphi_i(p_j)\right)_{1\leq i\leq d, 1\leq j\leq n}\,.$$

If

$$rk(\Phi) = d$$

and there exist $\boldsymbol{a} := (a_1,\dots,a_n) \in (\mathbb{R}^+)^n$ satisfying

$$\Phi\boldsymbol{a} = 0\,,$$

then there exist a cscK orbifold with isolated singularities $\left(\tilde{M},\tilde{g}\right)$ with a holomorphic surjective map

$$\pi : \tilde{M} \to M$$

and $\tilde{M}$ is obtained replacing points of $\boldsymbol{p}$ with resolutions X_p and points $\boldsymbol{q}$ with resolutions Y_q.

We give a proof of the above Theorem because this result and Theorem 1.7 are the first step in understanding if there is an “optimal” condition mixing different kind of orbifold singularities in order to get a cscK desingularization. In Chapter 5 of this thesis we look for examples in which Theorem 1.7 is applicable and we focus our research to Toric fano threefolds. We made this choice because all the conditions of Theorem 1.7 become combinatorial and hence easier to check. With the help of Gavin Brown and Alexander Kasprzyk that added in the Graded Ring Database [BK13],[Kas10] the classification of toric Fano threefolds with canonical singularities and told us how to use computer program MAGMA [BCP97] to recognize the type of singularities we restricted our research to six toric Fano orbifolds with canonical singularities and we identified the vertices of the moment polytope associated to $SU(3)$-singular cones. Unfortunately, at the moment, we can't say if we can apply Theorem 1.7 because we don't have sufficient knowledge of the Lie algebras of holomorphic vector fields vanishing somewhere of these varieties. We find explicit examples of orbifolds satisfying conditions (1.9) and (1.10) in dimension 2, and we need the extension of Theorem 1.7 to the case of surfaces. Our result extends verbatim, indeed, to dimension 2 but because of the technical complications due to the particular dimension we decided to give all the details in a forthcoming work and in Chapter 6 we focus only the examples we found. One of these examples is $\mathbb{P}^1 \times \mathbb{P}^1/\mathbb{Z}_2$ and it is discussed in [RS09a]. In [RS09a] the cscK metric on the crepant resolution of $\mathbb{P}^1 \times \mathbb{P}^1/\mathbb{Z}_2$ is constructed using the notion of parabolic polistability while here we can construct it by direct application of our gluing theory. In Chapter 6 we also discuss some conjectures and ideas for future work. A first natural continuation of our study is, in the spirit of the analysis of [Szé12] and [Szé13], the computation of the expansion of the Futaki invariant for the crepant resolutions of cscK orbifolds. This is the first step we can do to understand if there is a link, as in the case of blow ups of smooth cscK manifolds, with the notion of K-stability for orbifolds with cyclic singularities, that Ross and Thomas introduce in their work [RT11]. Another question that remains open is the following: “given a cscK orbifold is there a more general and unifying set of conditions relating orbifold points of different kind, smooth points and potentials of holomorphic vector fields that ensure the existence of a cscK resolution?” We conjecture that the answer to this question is positive but we don't know, at the moment, if the techniques we used in this thesis can be used to prove such a generalization because of the many technical difficulties that arise.

Brief outline of chapters content

In *Chapter 1* we recall some basic notions of complex and Kähler geometry and we introduce some technical results regarding cscK metrics that we will use intensively in successive chapters. We also explain in detail what kind of result we want to prove and the strategy of the proof. We warmly suggest to read section 1.6.2 where we give a detailed overview of the proof of Theorem 1.7.

In *Chapter 2* we investigate the properties of particular linear differential operators on cscK manifolds. More precisely we study their invertibility properties between weighted Hölder spaces.

In *Chapter 3* we begin the proof of our main result. With tools we introduced in chapter 2 we construct families, depending on some parameters, of cscK metrics on particular manifolds with boundary.

In *Chapter 4* we finish the proof we started in the preceding chapter. To conclude the proof we perform the connected sum construction along the boundaries of the manifolds we chose in chapter 3 and we glue the families of cscK metrics we constructed on them. To glue the families of metrics we use the technique known as Cauchy data matching. We also discuss the proof of Theorem 4.2.

In *Chapter 5* we look for examples of cscK orbifolds satisfying assumptions of Theorem 1.7. We focus our attention on toric 3-folds and it turns out that there is no toric three-dimensional orbifold satisfying our requests.

In *Chapter 6* we discuss the extension of Theorem 1.7 to 2-dimensional orbifolds and the relative technical issues. We discuss, moreover, some conjectures and ideas for future work.

Acknowledgements

The author wishes to thank Prof. Claudio Arezzo his for his guidance during his Ph.D., for suggesting him this problem and for his help regarding this thesis. He would also like to thank Dr. Lorenzo Mazzieri for his precious collaboration on the paper from which this thesis arises and Prof. Frank Pacard for helpful conversations and suggestions. Moreover he is grateful to Prof. Gavin Brown and Dr. Alexander Kasprzyk for the help given in a computation based on database of toric Fano varieties (GRDb). The author is indebted to the International School for Advanced Studies for funding his Ph.D. and supporting him in attending schools, workshops and conferences.

Contents

Chapter 1

The cscK Problem

In this chapter we will briefly recall notions of complex geometry and Kähler geometry that we will need. All the material of this chapter can be found in [Sim04],[Tia00].

1.1 A quick introduction to Kähler geometry

1.1.1 Complex Manifolds

We now briefly introduce complex manifolds and some of their properties. There are two possible definitions of complex manifold that turn out to be equivalent. The first definition is the most natural since it is very similar to the definition of a smooth real manifold.

Definition 1.1 (Complex manifolds 1)**.** Let M be a T_2 second countable topological space. M is a complex manifold if it has an atlas of open sets homeomorphic to open sets of $\mathbb{C}^m$ and the transition functions are holomorphic maps.

Now we give the second definition of complex manifold. Let M be a $2m$-dimensional smooth manifold, an almost-complex structure on M is a $J \in End\,(TM)$ s.t.

$$J^2 = -I\,.$$

Taking the complexification $TM_{\mathbb{C}} := TM \otimes \mathbb{C}$ and extending J $\mathbb{C}$-linearly to $J_{\mathbb{C}}$ we have that

$$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$$

with $T^{1,0}M, T^{0,1}M$ eigenbundles of $J_{\mathbb{C}}$. We say that J is a complex structure if $T^{0,1}M$ is an integrable distribution (in the sense of Frobenius).

Definition 1.2 (Complex manifolds 2)**.** Let M be $2m$-dimensional smooth manifold. M is a complex manifold if it has a complex structure.

The link between these two definitions is the famous theorem of Newlander-Nirenberg.

Theorem 1.1. *Let M be a smooth $2m$-dimensional manifold, M is a complex manifold if and only if it has a complex structure.*

If we have a complex structure, on a chart, we can choose coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ s.t.

$$J\left(\partial_{x_j}\right) = \partial_{y_j} \qquad J\left(\partial_{y_j}\right) = -\partial_{x_j}$$

and we can recover complex coordinates $(z_1, \ldots, z_m)$ putting

$$z_k = x_k + iy_k \,.$$

With these coordinates we have local bases of $T^{(1,0)}M$ and $(T^*)^{(1,0)}M$ given by

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right),$$
$$dz^k = dx^k + idy^k \,.$$

The complex structure gives a natural decomposition of complexified exterior powers of cotangent bundle

$$\Lambda^k_{\mathbb{C}}T^*M := \Lambda^k T^*M \otimes \mathbb{C}\,,$$

indeed

$$\Lambda^k_{\mathbb{C}}T^*M := \bigoplus_{p=0}^{k} \Lambda^{(p,k-p)}M$$

with $\Lambda^{(p,k-p)}M$ the subbundle of complex k-differential forms Ω of type

$$\Omega = \sum_{\substack{|I|=p \\ |J|=k-p}} \Omega_{IJ} dz^I \wedge \overline{dz^J}\,.$$

On complex manifolds the $\mathbb{C}$-linear extension of exterior differential d splits as a sum of two $\mathbb{C}$-linear operators

$$d = \partial + \overline{\partial}\,,$$
$$\overline{\partial} : \Lambda^{(p,q)}M \to \Lambda^{(p,q+1)}M\,,$$
$$\partial : \Lambda^{(p,q)}M \to \Lambda^{(p+1,q)}M\,,$$

with the following relations

$$\partial^2 = \overline{\partial}^2 = 0\,,$$
$$\partial\overline{\partial} = -\overline{\partial}\partial\,.$$

1.1.2 Kähler Manifolds

We now introduce a natural metric structure on complex manifolds: Hermitian metrics on $T^{1,0}M$ that are the analogue of Riemannian metrics on TM. We say that a metric g on TM is J-invariant if

$$g(\cdot,\cdot) = g(J\cdot, J\cdot)$$

that in local coordinates translates in

$$g_{ab} = J^c_a g_{cd} J^d_b\,.$$

If g is J-invariant we define the associated 2-form

$$\omega_g := g\left(\cdot, J\cdot\right) .$$

If we have a J-invariant metric we can consider its Hermitian extension to $TM_{\mathbb{C}}$ and its restriction h_g to $T^{1,0}M$ that becomes

$$h_g = \frac{1}{2}g + \frac{i}{2}\omega_g .$$

In complex coordinates we have

$$h_g = \sum_{\alpha,\beta=1}^{m} \left(h_g\right)_{\alpha\bar{\beta}} dz^\alpha \otimes \overline{dz^\beta} ,$$

$$\omega_g = \sum_{\alpha,\beta=1}^{m} i\left(h_g\right)_{\alpha\bar{\beta}} dz^\alpha \wedge \overline{dz^\beta} .$$

We now introduce the main object of our investigations: Kähler manifolds.

Definition 1.3. Let (M, g) a complex manifold with a J-invariant Riemannian metric g. (M, g) is a Kähler manifold if

$$d\omega_g = 0 .$$

The Kähler condition has many immediate implications, that we can summarize in the following proposition.

Proposition 1.1. *Let (M, g) be a Käher manifold and ∇ the Levi-Civita connection of g. Then*

- $\nabla J = 0$*;*
- *in holomorphic coordinates the following identities hold*

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z^\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial z^\alpha} \qquad \frac{\partial g_{\alpha\bar{\beta}}}{\partial \overline{z^\gamma}} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial \overline{z^\beta}} ;$$

- $\forall p \in M$ *there exist* $U(p)$ *and holomorphic coordinates such that*

$$\omega_g = \sum_{\alpha,\beta=1}^{m} \frac{i}{2}\delta_{\alpha\bar{\beta}} dz^\alpha \wedge \overline{dz^\beta} + iP_{\gamma\bar{\delta}\alpha\bar{\beta}} z^\gamma \overline{z^\beta} dz^\alpha \wedge \overline{dz^\beta} + \mathcal{O}\left(|z|^3\right) .$$

Taking the $\mathbb{C}$-linear extension $\nabla^{\mathbb{C}}$ of the Levi-Civita connection ∇ we can define the complex curvature tensor

$$R_{\mathbb{C}}\left(X, Y\right) Z = \nabla^{\mathbb{C}}_X \nabla^{\mathbb{C}}_Y Z - \nabla^{\mathbb{C}}_Y \nabla^{\mathbb{C}}_X Z - \nabla^{\mathbb{C}}_{[X,Y]} Z ,$$

and the complex Riemann tensor

$$R_{\mathbb{C}}\left(X, Y, Z, W\right) = g\left(R\left(X, Y\right) Z, W\right) .$$

We define also the Hermitian extension of Ricci tensor

$$\mathrm{Ric}^{\mathbb{C}}_g ,$$

and the Ricci 2-form

$$\rho_g = \mathrm{Ric}_g\left(\cdot, J\cdot\right) .$$

The properties of above objects can be summarized in the following proposition.

Proposition 1.2. *Let (M,g) a Kähler manifold, then in complex coordinates*

- *Christoffel symbols satisfy the following identities*

$$\overline{\Gamma^{\gamma}_{\alpha\beta}} = \Gamma^{\bar\gamma}_{\bar\alpha\bar\beta}\,,$$
$$\overline{\Gamma^{\bar\gamma}_{\alpha\beta}} = \Gamma^{\gamma}_{\bar\alpha\bar\beta} = \overline{\Gamma^{\bar\gamma}_{\alpha\bar\beta}} = \Gamma^{\gamma}_{\beta\bar\alpha} = 0\,,$$
$$\Gamma^{\gamma}_{\alpha\beta} = g^{\gamma\bar\delta}\partial_\beta g_{\alpha\bar\delta}\,;$$

- *the complex Riemann tensor satisfy the following identities*

$$R_{\alpha\bar\beta\gamma\bar\delta} = R_{\gamma\bar\beta\alpha\bar\delta} = R_{\alpha\bar\delta\gamma\bar\beta} = R_{\gamma\bar\delta\alpha\bar\beta} = \overline{R_{\beta\bar\alpha\delta\bar\gamma}} = -R_{\alpha\bar\beta\bar\delta\gamma}\,,$$
$$R_{\alpha\beta\gamma\bar\delta} = \overline{R_{\bar\alpha\bar\beta\delta\bar\gamma}} = 0\,,$$
$$R_{\alpha\bar\beta\gamma\bar\delta} = -\partial_\gamma\overline{\partial}_\delta g_{\alpha\bar\beta} + g^{\epsilon\bar\phi}\partial_\gamma g_{\epsilon\bar\beta}\overline{\partial}_\delta g_{\alpha\bar\phi}\,,$$
$$\nabla_\epsilon R_{\alpha\bar\beta\gamma\bar\delta} = \nabla_\alpha R_{\epsilon\bar\beta\gamma\bar\delta}$$

- *The Ricci tensor and Ricci form satisfy the following identities*

$$\overline{Ric_{\alpha\beta}} = Ric_{\bar\alpha\bar\beta} = 0\,,$$
$$\overline{Ric_{\beta\bar\alpha}} = Ric_{\alpha\bar\beta} = g^{\gamma\bar\delta}R_{\alpha\bar\beta\gamma\bar\delta} = -\partial_\alpha\overline{\partial}_\beta\log\left(\det\left(g_{\gamma\bar\delta}\right)\right)\,,$$
$$\nabla_\gamma Ric_{\alpha\bar\beta} = \nabla_\alpha Ric_{\gamma\bar\beta} \qquad \nabla_{\bar\delta} Ric_{\alpha\bar\beta} = \nabla_{\bar\beta} Ric_{\gamma\bar\delta}\,,$$
$$\rho = \frac{i}{2}Ric_{\alpha\bar\beta}dz^\alpha\wedge\overline{dz^\beta}\,;$$

- *the scalar curvature s_g can be computed by the formula*

$$s_g = 2g^{\alpha\bar\beta}Ric_{\alpha\bar\beta} = -2g^{\alpha\bar\beta}\partial_\alpha\overline{\partial}_\beta\log\left(\det\left(g_{\gamma\bar\delta}\right)\right)\,. \tag{1.1}$$

Moreover for each $p\in M$ there exist complex coordinates (Kähler normal coordinates) such that the metric g and the Ricci tensor Ric can be written in the following way

$$g_{\alpha\bar\beta} = \partial_\alpha\overline{\partial}_\beta\left[\frac{|z|^2}{2} - \frac{1}{4}R_{\alpha\bar\beta\gamma\bar\delta}(p)z^\alpha\overline{z^\beta}z^\gamma\overline{z^\delta} + \mathcal{O}(|z|^5)\right] = \frac{1}{2}\delta_{\alpha\bar\beta} - R_{\alpha\bar\beta\gamma\bar\delta}(p)z^\gamma\overline{z^\delta} + \mathcal{O}(|z|^3)\,,$$
$$Ric_{\alpha\bar\beta} = \mu_\alpha\delta_{\alpha\bar\beta} + \mathcal{O}(|z|)\,.$$

We introduce Laplace operator on functions as

$$\Delta_g = 2g^{\alpha\bar\beta}\partial_\alpha\bar\partial_\beta$$

and more generally on tensor of any type using Levi-Civita connection

$$\Delta_g = g^{\alpha\bar\beta}\left(\nabla_\alpha\nabla_{\bar\beta} + \nabla_{\bar\beta}\nabla_\alpha\right)\,.$$

1.2 The flat case

Let's consider a small open ball $B_\varepsilon\subseteq\mathbb{C}^m$, that is the local model of any Kähler manifold of dimension m. Since it is a subset of the complex euclidean space, B_ε carries naturally a (non complete) Kähler metric g_0 with associated Kähler form ω_0 that comes from the euclidean one. Suppose we want to find a small perturbation of the euclidean metric such that its scalar curvature is constant. Making things explicit, we want to find functions $f \in C^4(B_\varepsilon)$ such that on B_ε

$$\omega_0 + i\partial\overline{\partial} f = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + f\right)$$

is positive and

$$s_{\omega_0 + i\partial\overline{\partial} f} \equiv \sigma \quad \sigma \in \mathbb{R}\,.$$

For a Kähler manifold we have an explicit formula for calculating the scalar curvature of a metric in a coordinate chart, indeed, if $g_{i\bar{j}}$ is the metric tensor, we have formula (1.1)

$$s_g = -2g^{i\bar{j}}\partial_j\overline{\partial}_i \log\left(\det\left(g_{a\bar{b}}\right)\right)\,.$$

From now on, for every tensor of type

$$T = T_{i\bar{j}} dz_i \otimes d\bar{z}_j\,,$$

on a complex manifold M and every $u \in \mathcal{C}^\infty(M)$ we indicate

$$\operatorname{tr}\left(\partial\overline{\partial}u \cdot T\right) = g^{i\bar{l}} g^{k\bar{j}} \partial_i \overline{\partial}_j u\, T_{k\bar{l}}.$$

In our case $g_{i\bar{j}} = \delta_{i\bar{j}}$, if we put $\mathbb{I}_m = \delta_{i\bar{j}}$ and $F = \partial_i\overline{\partial}_j f$ the formula becomes

$$s_{\omega_0 + i\partial\overline{\partial} f} = -4\operatorname{tr}\left(\left(\mathbb{I}_m + 2F\right)^{-1} \cdot \partial\overline{\partial}\log\left(\det\left(\mathbb{I}_m + 2F\right)\right)\right)\,.$$

Now we have a (nonlinear) operator

$$\begin{gathered}\mathbf{s}_0 : D \subseteq C^4(B_\varepsilon) \to C^0(B_\varepsilon)\,,\\ \mathbf{s}_0(f) = s_{\omega_0 + i\partial\overline{\partial} f}\,.\end{gathered}$$

To understand better the operator $\mathbf{s}_0$ we want to decompose it in simpler pieces, and to do it we have to make some assumptions.

Lemma 1.1. *Let $f \in C^4(B_\varepsilon)$ and s.t. $\|f\|_{C^4(B_\varepsilon(0))} \leq C(\varepsilon)$ with $C(\varepsilon) \in \mathbb{R}$ sufficiently small; then if we set*

$$\partial\overline{\partial} f = \left(\partial_i\overline{\partial}_j f\right)_{1\leq i,j\leq m}\,,$$

we have the following formula

$$\mathbf{s}_0(f) = -\frac{\Delta^2 f}{2} + 4tr\left(\partial\overline{\partial} f \cdot \partial\overline{\partial}\Delta f\right) + 2\Delta\, tr\left(\left(\partial\overline{\partial} f\right)^2\right) + Q_0^3(f)\,, \tag{1.2}$$

with Q_0^3 a nonlinear analytic function of degree at least 3 on the derivatives of order $2, 3, 4$ of f.

Proof. The nonlinear operator $\mathbf{s}_0$ has analytic dependence on its arguments, and for small f we have

$$\mathbf{s}_0(f) = \sum_{k=0}^{+\infty} \frac{1}{k!}\mathbf{s}_0^k(f)\,, \tag{1.3}$$

with $\mathbf{s}_0^k(f)$ a homogeneous polynomial of degree k in f and its derivatives. To get the above decomposition we consider $\mathbf{s}_0(tf)$ with $t \in (-\tau, \tau)$ with $\tau << 1$ and we set

$$\mathbf{s}_0^k(f) := \left.\frac{d^k}{dt^k}\mathbf{s}_0(tf)\right|_{t=0} .$$

We recall that

$$\mathbf{s}_0(tf) = -4\mathrm{tr}\left((\mathbb{I}_m + 2tF)^{-1} \cdot \partial\overline{\partial}\log\left(\det\left(\mathbb{I}_m + 2tF\right)\right)\right) .$$

Expanding we have

$$\begin{aligned}\mathbf{s}_0(tf) &= -4\mathrm{tr}\left((\mathbb{I}_m + 2tF)^{-1} \cdot \partial\overline{\partial}\log\left(\det\left(\mathbb{I}_m + 2tF\right)\right)\right)\\ &= -4\mathrm{tr}\left((\mathbb{I}_m + 2tF)^{-1} \cdot \partial\overline{\partial}\log\left(1 + \sum_{k=1}^{m} 2^k t^k \sigma_k(F)\right)\right)\end{aligned}$$

with σ_k the k - th symmetric function on the eigenvalues of a matrix. So we have

$$\mathbf{s}_0^k(f) = -4\sum_{l=1}^{k}\mathrm{tr}\left[\frac{d^{(k-l)}}{dt^{(k-l)}}(\mathbb{I}_m + 2tF)^{-1} \cdot \partial\overline{\partial}\frac{d^l}{dt^l}\log\left(1 + \sum_{h=1}^{m} 2^h t^h \sigma_h(F)\right)\right] .$$

By its very definition we have that

$$\mathbf{s}_0^k(f) = P_{1,k-1}\left(\nabla^2 F, F\right) + P_{2,k-2}\left(\nabla F, F\right) ,$$

with $P_{a,b}$ homogeneous polynomials on $\mathbb{C}^{2m^2}$ of degree $a+b$ with constant coefficients and of degree a in the first m^2 variables and of degree b in the other m^2 variables. Now we compute exactly $\mathbf{s}_0^1(f)$, $\mathbf{s}_0^2(f)$, $\mathbf{s}_0^3(f)$. For the sake of notation we define

$$\begin{aligned}\Upsilon = \Upsilon(t,F) &:= \sum_{k=1}^{m} 2^k t^k \sigma_k(F) ,\\ \Xi = \Xi(t,F) &:= (\mathbb{I}_m + 2tF)^{-1} ,\end{aligned}$$

$$\mathbf{s}_0(tf) = -4\mathrm{tr}\left(\Xi \cdot \partial\overline{\partial}\log(1+\Upsilon)\right) .$$

We start with $\mathbf{s}_0^1(f)$.

$$\frac{d}{dt}\mathbf{s}_0(tf) = -4\mathrm{tr}\left(\Xi \cdot \partial\overline{\partial}\frac{\partial_t \Upsilon}{(1+\Upsilon)}\right) + 8\mathrm{tr}\left(\Xi \cdot F \cdot \Xi \cdot \partial\overline{\partial}\log(1+\Upsilon)\right) .$$

Evaluating at $t=0$ we have

$$\mathbf{s}_0^1(f) = \left.\frac{d}{dt}\mathbf{s}_0(tf)\right|_{t=0} = -\frac{\Delta^2 f}{2} .$$

Now we compute $\mathbf{s}_0^2(f)$.

$$\begin{aligned}\frac{d^2}{dt^2}\mathbf{s}_0(tf) &= -4\mathrm{tr}\left(\Xi \cdot \partial\overline{\partial}\partial_t\frac{\partial_t \Upsilon}{(1+\Upsilon)}\right) + 16\mathrm{tr}\left(\Xi \cdot F \cdot \Xi \cdot \partial\overline{\partial}\frac{\partial_t \Upsilon}{(1+\Upsilon)}\right)\\ &\quad - 32\mathrm{tr}\left(\Xi \cdot F \cdot \Xi \cdot F \cdot \Xi \cdot \partial\overline{\partial}\log(1+\Upsilon)\right)\\ &= -4\mathrm{tr}\left(\Xi \cdot \partial\overline{\partial}\frac{\partial_t^2 \Upsilon}{(1+\Upsilon)}\right) + 4\mathrm{tr}\left(\Xi \cdot \partial\overline{\partial}\frac{(\partial_t \Upsilon)^2}{(1+\Upsilon)^2}\right)\\ &\quad + 16\mathrm{tr}\left(\Xi \cdot F \cdot \Xi \cdot \partial\overline{\partial}\frac{\partial_t \Upsilon}{(1+\Upsilon)}\right) - 32\mathrm{tr}\left(\Xi \cdot F \cdot \Xi \cdot F \cdot \Xi \cdot \partial\overline{\partial}\log(1+\Upsilon)\right)\end{aligned}$$

and evaluating at $t = 0$ we have

$$\begin{aligned}
\mathbf{s}_0^2(f) &= \left.\frac{d^2}{dt^2}\mathbf{s}_0(tf)\right|_{t=0} \\
&= -32\mathrm{tr}\left(\partial\overline{\partial}\sigma_2(F)\right) + 16\mathrm{tr}\left(\partial\overline{\partial}\sigma_1(F)^2\right) + 32\mathrm{tr}\left(F\cdot\partial\overline{\partial}\sigma_1(F)\right) \\
&= -16\mathrm{tr}\left(\partial\overline{\partial}\mathrm{tr}(F)^2\right) + 16\mathrm{tr}\left(\partial\overline{\partial}\mathrm{tr}\left(F^2\right)\right) + 16\mathrm{tr}\left(\partial\overline{\partial}\mathrm{tr}(F)^2\right) + 32\mathrm{tr}\left(F\cdot\partial\overline{\partial}\mathrm{tr}(F)\right) \\
&= 32\mathrm{tr}\left(F\cdot\partial\overline{\partial}\mathrm{tr}(F)\right) + 16\mathrm{tr}\left(\partial\overline{\partial}\mathrm{tr}\left(F^2\right)\right) \\
&= 8\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\Delta f\right) + 4\Delta\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^2\right).
\end{aligned}$$

Arguing analogously we find that

$$\mathbf{s}_0^3(f) = -12\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^2\cdot\partial\overline{\partial}\Delta f\right) - 48\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^2\right)\right) - 16\Delta\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^3\right).$$

Now setting

$$Q_0^3(f) := -2\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^2\cdot\partial\overline{\partial}\Delta f\right) - 8\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^2\right)\right) - \frac{8}{3}\Delta\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^3\right) + \sum_{k=4}^{+\infty}\frac{1}{k!}\mathbf{s}_0^k(f),$$

we have the desired decomposition.

□

Suppose we are looking for a small perturbation $f \in C^4(B_\varepsilon)$, of the euclidean potential such that the resulting metric has constant scalar curvature. We can use the expansion in Lemma 1.1 to get the differential equation that f has to satisfy

$$-\frac{\Delta^2 f}{2} = \sigma - 4\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\Delta f\right) - 2\Delta\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^2\right) - Q_0^3(f),$$

with $\sigma \in \mathbb{R}$. Solutions to this equation have a nice regularity property, indeed they are real analytic.

Proposition 1.3. *Let $f \in C^4(B_\varepsilon)$, with $\|f\|_{C^4(B_\varepsilon)} \le C(\varepsilon)$ with $C(\varepsilon)$ sufficiently small be a solution of*

$$-\frac{\Delta^2 f}{2} = \sigma - 4tr\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\Delta f\right) - 2\Delta tr\left(\left(\partial\overline{\partial}f\right)^2\right) - Q_0^3(f)$$

with $\sigma \in \mathbb{R}$. We have then $f \in C^\omega(B_\varepsilon)$.

Proof. Let's consider the equation

$$-\frac{\Delta^2 f}{2} = \sigma - 4\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\Delta f\right) - 2\Delta\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^2\right) - Q_0^3(f).$$

It is elliptic since its linear part Δ^2 is strongly elliptic, and its nonlinear part, the right hand side, is an analytic function of degree at least 2 on the solution and its derivatives up to the order 4. So by the regularity theory developed in [Mor58], we have that $f \in C^\omega(B_\varepsilon)$ □

We do not explore, for the moment, the problem of existence of the solution for that equation; we only want to keep in mind its main features that are

- ellipticity,
- the analytic dependence on its solutions,
- the regularity of its solutions.

What we want to do now is to globalize this problem: our domain will be a whole Kähler manifold (M, g) and we want to perturb its metric g with a "small function $f \in C^4(M)$" such that the resulting metric has constant scalar curvature.

1.3 The scalar curvature operator on a Kähler manifold

The first step to go on with our study is to analyze how changes the scalar curvature of M when we perturb g with a function $f \in C^4(M)$. Obviously the function f with which we want to modify g has to satisfy the condition

$$\omega_g + i\partial\overline{\partial} f > 0\,,$$

on the whole M. First of all we want to have a "global" analogous of Lemma 1.1, and this will require some work. We suppose that $\|f\|_{C^4(M)} < C$ with $C \in \mathbb{R}^+$ sufficiently small; we want to calculate variations of

$$\mathbf{s}_g(f) := s_{\omega_g + i\partial\overline{\partial} f}\,.$$

If we are given a function $f \in C^2(M)$, we can associate to f a continuous endomorphism of the holomorphic tangent bundle $\mathcal{D}(f) \in C^0\left(\Omega^{(1,0)}M \otimes T^{(1,0)}M\right)$ in this way: we pick coordinate charts and we define

$$\mathcal{D}(f) := \partial_i \overline{\partial}_j f g^{k\bar{j}} dz^i \otimes \frac{\partial}{\partial z^k}\,.$$

Clearly this local formula is well defined and so $\mathcal{D}(f) \in C^0\left(\Omega^{(1,0)}M \otimes T^{(1,0)}M\right)$. We recall once again that in local coordinates we have formula (1.1) for scalar curvature that is

$$s_g = -2g^{i\bar{j}}\partial_i\partial_{\bar{j}} \log\left(\det\left(g_{a\bar{b}}\right)\right)\,.$$

If we perturb the metric $g_{i\bar{j}}$ with $\partial_i\partial_{\bar{j}} f$ with $f \in C^4(M)$ and we set

$$\tilde{g}_{i\bar{j}} := g_{i\bar{j}} + \partial_i\partial_{\bar{j}} f\,,$$

we have

$$\mathbf{s}_g(f) = -2\tilde{g}^{i\bar{j}}\partial_i\partial_{\bar{j}} \log\left(\det\left(\tilde{g}_{a\bar{b}}\right)\right)\,.$$

We have the following identity

$$\begin{aligned}
\tilde{g}^{a\bar{i}}\tilde{g}_{a\bar{j}} = \delta_{\bar{j}}^{\bar{i}} &\Rightarrow \tilde{g}^{a\bar{i}} g_{a\bar{j}} + \tilde{g}^{a\bar{i}}\partial_a\overline{\partial}_j f = \delta_{\bar{j}}^{\bar{i}} \\
&\Rightarrow \tilde{g}^{a\bar{i}}\left[\delta_a^b + g^{b\bar{c}}\partial_a\overline{\partial}_c f\right] = g^{i\bar{b}} \\
&\Rightarrow \left(\mathbb{I}_m + \mathcal{D}(f)\right)_a^b \tilde{g}^{a\bar{i}} = g^{i\bar{b}} \\
&\Rightarrow \tilde{g}^{i\bar{j}} = \left[\left(\mathbb{I}_m + \mathcal{D}(f)\right)^{-1}\right]_a^i g^{a\bar{j}}
\end{aligned}$$

with $\mathbb{I}_m$ the identity endomorphism in $\Omega^{(1,0)}M \otimes T^{(1,0)}M$. So we have

$$
\begin{aligned}
\mathbf{s}(f) =& -2\tilde{g}^{i\bar{j}}\partial_i\partial_{\bar{j}}\log\left(\det\left(\tilde{g}_{b\bar{c}}\right)\right)\\
=& -2\left[\left(\mathbb{I}_m+\mathcal{D}\left(f\right)\right)^{-1}\right]_a^i g^{a\bar{j}}\partial_i\partial_{\bar{j}}\log\left(\det\left(g_{b\bar{c}}+\partial_b\overline{\partial}_c f\right)\right)\\
=& -2\left[\left(\mathbb{I}_m+\mathcal{D}\left(f\right)\right)^{-1}\right]_a^i g^{a\bar{j}}\partial_i\partial_{\bar{j}}\log\left(\det\left(g_{b\bar{c}}\right)\right)\\
& -2\left[\left(\mathbb{I}_m+\mathcal{D}\left(f\right)\right)^{-1}\right]_a^i g^{a\bar{j}}\partial_i\partial_{\bar{j}}\log\left(\det\left(\delta_b^d+g^{d\bar{c}}\partial_b\overline{\partial}_c f\right)\right)\\
=& 2\left[\left(\mathbb{I}_m+\mathcal{D}\left(f\right)\right)^{-1}\right]_a^i \mathrm{Ric}_i^a-2\left[\left(\mathbb{I}_m+\mathcal{D}\left(f\right)\right)^{-1}\right]_a^i\left[\mathcal{D}\left(\log\left(\det\left(\mathbb{I}_m+\mathcal{D}\left(f\right)\right)\right)\right)\right]_i^a\\
=& 2\mathrm{tr}\left[\left(\mathbb{I}_m+\mathcal{D}\left(f\right)\right)^{-1}\mathrm{Ric}^\sharp\right]-2\mathrm{tr}\left[\left(\mathbb{I}_m+\mathcal{D}\left(f\right)\right)^{-1}\mathcal{D}\left(\log\left(\det\left(\mathbb{I}_m+\mathcal{D}\left(f\right)\right)\right)\right)\right]
\end{aligned}
$$

with

$$
\mathrm{Ric}^\sharp=\mathrm{Ric}_{i\bar{j}}g^{k\bar{j}}dz^i\otimes\frac{\partial}{\partial z^k}
$$

and $\mathrm{tr}(\cdot)$ the trace of an endomorphism. Now that we have a coordinate free expression in terms of f for $\mathbf{s}_g(f)$, we can look for a global analogue of the Lemma 1.1.

Lemma 1.2. *Let $f\in C^4(M)$ with $\|f\|_{C^4(M)}<C$ for sufficiently small $C\in\mathbb{R}^+$ then we can expand the operator $\mathbf{s}_g(f)$ in the following way*

$$
\mathbf{s}_g(f)=s_g-\mathbb{L}_g f+Q_g\left(f\right)\,,
$$

with

$$
\mathbb{L}_g f=\frac{\Delta_g^2 f}{2}+2tr\left(\mathcal{D}\left(f\right)Ric^\sharp\right)=\frac{\Delta_g^2 f}{2}+2\left\langle Ric_g,i\partial\overline{\partial}f\right\rangle_g\,,
$$

$$
Q_g\left(f\right)=tr\left(\mathcal{D}\left(f\right)^2 Ric^\sharp\right)-tr\left(\mathcal{D}\left(f\right)\mathcal{D}\left(\Delta_g f\right)\right)+\frac{1}{2}\Delta_g tr\left(\mathcal{D}\left(f\right)^2\right)+Q_g^3\left(f\right)\,,
$$

and Q_g^3 an analytic function of degree at least 3 on $\mathcal{D}\left(f\right),\nabla\mathcal{D}\left(f\right),\nabla^2\mathcal{D}\left(f\right)$.

Proof. As in the euclidean case we have that $\mathbf{s}_g\left(f\right)$ is an analytic function on its arguments so for small f we have

$$
\mathbf{s}_g\left(f\right)=\sum_{k=0}^{+\infty}\frac{1}{k!}\mathbf{s}_g^k\left(f\right)
$$

with $\mathbf{s}_g^k\left(f\right)$ a homogeneous polynomial of degree k in $\mathcal{D}\left(f\right)$ and its covariant derivatives. To get the above decomposition we consider $\mathbf{s}_g\left(tf\right)$ with $t\in(-\tau,\tau)$ with $\tau<<1$ and we set

$$
\mathbf{s}_g^k\left(f\right):=\left.\frac{d^k}{dt^k}\mathbf{s}_g\left(tf\right)\right|_{t=0}\,.
$$

We want to compute exactly $\mathbf{s}_g^1(f),\mathbf{s}_g^1(f)$. To do this we take $t\in(-\tau,\tau)$ and we compute

$$
\left.\frac{d}{dt}\mathbf{s}\left(tf\right)\right|_{t=0}
$$

and

$$\left.\frac{d^2}{dt^2}\mathbf{s}\left(tf\right)\right|_{t=0} .$$

The first derivative gives

$$\begin{aligned}\frac{d}{dt}\mathbf{s}\left(tf\right) =& 2\sum_{k=1}^{+\infty}\left(-1\right)^k kt^{k-1}\mathrm{tr}\left(\mathcal{D}\left(f\right)^k \mathrm{Ric}^\sharp\right)\\ &- 2\sum_{k=1}^{+\infty}\left(-1\right)^k kt^{k-1}\mathrm{tr}\left(\mathcal{D}\left(f\right)^k \mathcal{D}\left(\log\left(\det\left(\mathbb{I}_m + t\mathcal{D}\left(f\right)\right)\right)\right)\right)\\ &- 2\sum_{k=1}^{m} kt^{k-1}\mathrm{tr}\left(\left(\mathbb{I}_m + t\mathcal{D}\left(f\right)\right)^{-1}\mathcal{D}\left(\frac{\sigma_k\left(\mathcal{D}\left(f\right)\right)}{\det\left(\mathbb{I}_m + t\mathcal{D}\left(f\right)\right)}\right)\right)\end{aligned}$$

and evaluating at $t = 0$ we obtain

$$\begin{aligned}\frac{d}{dt}\mathbf{s}\left(tf\right)|_{t=0} =& -2\mathrm{tr}\left(\mathcal{D}\left(f\right)\mathrm{Ric}^\sharp\right) - 2\mathrm{tr}\left(\mathcal{D}\left(\sigma_1\left(\mathcal{D}\left(f\right)\right)\right)\right)\\ =& -\frac{\Delta_g^2 f}{2} - 2\left\langle \mathrm{Ric}_g, i\partial\overline{\partial}f\right\rangle_g .\end{aligned}$$

Differentiating again we have

$$\begin{aligned}\frac{d^2}{dt^2}\mathbf{s}\left(tf\right) =& 2\sum_{k=2}^{+\infty}\left(-1\right)^k k(k-1)t^{k-2}\mathrm{tr}\left(\mathcal{D}\left(f\right)^k \mathrm{Ric}^\sharp\right)\\ &- 2\sum_{k=2}^{+\infty}\left(-1\right)^k k(k-1)t^{k-2}\mathrm{tr}\left(\mathcal{D}\left(f\right)^k \mathcal{D}\left(\log\left(\det\left(\mathbb{I}_m + t\mathcal{D}\left(f\right)\right)\right)\right)\right)\\ &- 4\sum_{k=1}^{m}\sum_{l=1}^{+\infty}\left(-1\right)^l klt^{l+k-2}\mathrm{tr}\left(\mathcal{D}\left(f\right)^l \mathcal{D}\left(\frac{\sigma_k\left(\mathcal{D}\left(f\right)\right)}{\det\left(\mathbb{I}_m + t\mathcal{D}\left(f\right)\right)}\right)\right)\\ &- 2\sum_{k=2}^{m} k(k-1)t^{k-2}\mathrm{tr}\left(\left(\mathbb{I}_m + t\mathcal{D}\left(f\right)\right)^{-1}\mathcal{D}\left(\frac{\sigma_k\left(\mathcal{D}\left(f\right)\right)}{\det\left(\mathbb{I}_m + t\mathcal{D}\left(f\right)\right)}\right)\right)\\ &+ 2\sum_{k,l=1}^{m} klt^{k+l-2}\mathrm{tr}\left(\left(\mathbb{I}_m + t\mathcal{D}\left(f\right)\right)^{-1}\mathcal{D}\left(\frac{\sigma_k\left(\mathcal{D}\left(f\right)\right)\sigma_l\left(\mathcal{D}\left(f\right)\right)}{\det\left(\mathbb{I}_m + t\mathcal{D}\left(f\right)\right)^2}\right)\right)\end{aligned}$$

and evaluating at $t = 0$

$$\begin{aligned}\frac{d^2}{dt^2}\mathbf{s}\left(tf\right)|_{t=0} =& 4\mathrm{tr}\left(\mathcal{D}\left(f\right)^2 Ric^\sharp\right) + 4\mathrm{tr}\left(\mathcal{D}\left(f\right)\mathcal{D}\left(\sigma_1\left(\mathcal{D}\left(f\right)\right)\right)\right)\\ &- 4\mathrm{tr}\left(\mathcal{D}\left(\sigma_2\left(\mathcal{D}\left(f\right)\right)\right)\right) + 2\mathrm{tr}\left(\mathcal{D}\left(\sigma_1\left(\mathcal{D}\left(f\right)\right)\right)^2\right)\\ =& 4\mathrm{tr}\left(\mathcal{D}\left(f\right)^2 Ric^\sharp\right) + 4\mathrm{tr}\left(\mathcal{D}\left(f\right)\mathcal{D}\left(\mathrm{tr}\left(\mathcal{D}\left(f\right)\right)\right)\right)\\ &+ 2\mathrm{tr}\left(\mathcal{D}\left(\mathrm{tr}\left(\mathcal{D}\left(f\right)^2\right)\right)\right)\\ =& 4\mathrm{tr}\left(\mathcal{D}\left(f\right)^2 Ric^\sharp\right) + 2\mathrm{tr}\left(\mathcal{D}\left(f\right)\mathcal{D}\left(\Delta_g f\right)\right)\\ &+ \Delta_g\mathrm{tr}\left(\mathcal{D}\left(f\right)^2\right) .\end{aligned}$$

If we set

$$Q_g^3(f) := \sum_{k=3}^{+\infty} \frac{1}{k!} \mathbf{s}_g^k(f) \ ,$$

then we have the decomposition stated in the lemma. □

Let (M, g) be a Kähler manifold, suppose we are in a coordinate chart and suppose that the metric g is the euclidean one perturbed by a "small" function. With the following lemma we will show that, on that chart, the scalar curvature operator $\mathbf{s}_g$ is very close to $\mathbf{s}_0$, the one of the flat case.

Lemma 1.3. *Let (M, g) be a Kähler manifold, U an open coordinate set, and let on this set*

$$\omega_g = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + \psi_g(z)\right) = \omega_0 + i\partial\overline{\partial}\psi_g \ ,$$

with $\psi_g \in C^4(U)$ and $\|\psi_g\|_{C^4(U)} \leq C_g$ with C_g small. Let moreover $f \in C^4(M)$ and $\|f\|_{C^4(M)} \leq C$ with C sufficiently small, then we have on U

$$\boldsymbol{s}_g(f) - s_g - \boldsymbol{s}_0(f) = \widetilde{\mathbb{L}}_g f + \tilde{Q}_g^2(f) \ ,$$

with

$$\widetilde{\mathbb{L}}_g := \mathbb{L}_g - \frac{\Delta^2}{2}$$

a linear operator with coefficients depending at least linearly on ψ_g and its derivatives and $\tilde{Q}_g^2$ a function with dependence at least quadratic on f and its derivatives and at least linear dependence on ψ_g and its derivatives.

Proof. The proof of this lemma amounts to computations in a coordinate chart. We note that

$$\mathbf{s}_g(f) = \mathbf{s}_0(\psi_g + f) \ ,$$

so we have to compute the quantity

$$\mathbf{s}_g(f) - s_g - \mathbf{s}_0(f) = \mathbf{s}_0(\psi_g + f) - \mathbf{s}_0(\psi_g) - \mathbf{s}_0(f) \ .$$

By Lemma 1.1, in particular, using relation (1.3) we have

$$\mathbf{s}_0(\psi_g + f) - \mathbf{s}_0(\psi_g) - \mathbf{s}_0(f) = \sum_{k=1}^{+\infty} \frac{1}{k!}\left(\mathbf{s}_0^k(\psi_g + f) - \mathbf{s}_0^k(\psi_g) - \mathbf{s}_0^k(f)\right)$$

and so

$$
\begin{aligned}
\mathbf{s}_0^k\left(\psi_g+f\right)-\mathbf{s}_0^k\left(\psi_g\right)-\mathbf{s}_0^k\left(f\right)=&P_{1,k-1}\left(\nabla^4\left(\psi_g+f\right),\nabla^2\left(\psi_g+f\right)\right)\\
&+P_{2,k-2}\left(\nabla^3\left(\psi_g+f\right),\nabla^2\left(\psi_g+f\right)\right)\\
&-P_{1,k-1}\left(\nabla^4\psi_g,\nabla^2\psi_g\right)-P_{2,k-2}\left(\nabla^3\psi_g,\nabla^2\psi_g\right)\\
&-P_{1,k-1}\left(\nabla^4 f,\nabla^2 f\right)-P_{2,k-2}\left(\nabla^3 f,\nabla^2 f\right)\\
=&\sum_{l=0}^{k-1}A_{1,0,l,k-1-l}\left(\nabla^4 f,\nabla^4\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+\sum_{l=0}^{k-1}A_{0,1,k-1-l,l}\left(\nabla^4 f,\nabla^4\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+\sum_{l=1}^{k-1}A_{0,0,k-l,l}\left(\nabla^4 f,\nabla^4\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+\sum_{l=1}^{k-2}B_{2,0,k-2-l,l}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+\sum_{l=1}^{k-2}B_{0,2,l,k-2-l}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+\sum_{l=0}^{k-2}B_{1,1,k-2-l,l}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+\sum_{l=1}^{k-1}B_{1,0,k-1-l,l}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+\sum_{l=1}^{k-1}B_{0,1,k-1-l,l}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+\sum_{l=1}^{k-1}B_{0,0,k-l,l}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)
\end{aligned}
$$

with $A_{d_1,d_2,d_3,d_4}, B_{d_1,d_2,d_3,d_4}$ polynomials with constant coefficients of degree $d_1+d_2+d_3+d_4$ and of degree d_i in the i-th entry.

We define

$$
\begin{aligned}
L_k(f)=&A_{1,0,0,k-1}\left(\nabla^4 f,\nabla^4\psi_g,\nabla^2 f,\nabla^2\psi_g\right)+A_{0,1,1,k-2}\left(\nabla^4 f,\nabla^4\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+A_{0,0,1,k-1}\left(\nabla^4 f,\nabla^4\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+B_{0,2,1,k-3}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)+B_{1,1,0,k-2}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+B_{1,0,0,k-1}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)+B_{0,1,1,k-2}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\\
&+B_{0,0,1,k-1}\left(\nabla^3 f,\nabla^3\psi_g,\nabla^2 f,\nabla^2\psi_g\right)\,.
\end{aligned}
$$

We now calculate explicitly the linear part in f of $\mathbf{s}_0\left(\psi_g+f\right)-\mathbf{s}_0\left(\psi_g\right)-\mathbf{s}_0\left(f\right)$ that is

$$
\widetilde{\mathbb{L}}_g f:=-\left.\frac{d}{dt}\left[\mathbf{s}_0\left(\psi_g+tf\right)-\mathbf{s}_0\left(tf\right)\right]\right|_{t=0}\,.
$$

Using Lemma 1.1

$$
\begin{aligned}
\widetilde{\mathbb{L}}_g f =& - \frac{d}{dt}\left[\mathbf{s}_0\left(\psi_g + tf\right) - \mathbf{s}_0\left(tf\right)\right]\Big|_{t=0} \\
=& - \frac{d}{dt}\left[\mathbf{s}_0^1\left(\psi_g + tf\right) - \frac{1}{2}\mathbf{s}_0^2\left(\psi_g + tf\right) + \sum_{k=3}^{+\infty}\frac{1}{k!}\mathbf{s}_0^k\left(\psi_g + tf\right) - \mathbf{s}_0^1\left(tf\right) + \sum_{k=2}^{+\infty}\frac{1}{k!}\mathbf{s}_0^k\left(tf\right)\right]\Bigg|_{t=0} \\
=& \frac{1}{2}\Delta^2 f - \frac{d}{dt}\left[\frac{1}{2}\mathbf{s}_0^2\left(\psi_g + tf\right)\right]\Big|_{t=0} - \frac{d}{dt}\left[\sum_{k=3}^{+\infty}\frac{1}{k!}\mathbf{s}_0^k\left(\psi_g + tf\right)\right]\Bigg|_{t=0} - \frac{1}{2}\Delta^2 f \\
=& - \frac{d}{dt}\left[\frac{1}{2}\mathbf{s}_0^2\left(\psi_g + tf\right)\right]\Big|_{t=0} - \frac{d}{dt}\left[\sum_{k=3}^{+\infty}\frac{1}{k!}\mathbf{s}_0^k\left(\psi_g + tf\right)\right]\Bigg|_{t=0} \\
=& - \frac{1}{2}\frac{d}{dt}\left[8\mathrm{tr}\left(\partial\overline{\partial}\left(\psi_g + tf\right)\cdot\partial\overline{\partial}\Delta\left(\psi_g + tf\right)\right) + 4\Delta\mathrm{tr}\left(\left(\partial\overline{\partial}\left(\psi_g + tf\right)\right)^2\right)\right]\Big|_{t=0} \\
& - \frac{d}{dt}\left[\sum_{k=3}^{+\infty}\frac{1}{k!}\mathbf{s}_0^k\left(\psi_g + tf\right)\right]\Bigg|_{t=0} \\
=& - \frac{d}{dt}\left[4\mathrm{tr}\left(\partial\overline{\partial}\left(\psi_g + tf\right)\cdot\partial\overline{\partial}\Delta\left(\psi_g + tf\right)\right) + 2\Delta\mathrm{tr}\left(\left(\partial\overline{\partial}\left(\psi_g + tf\right)\right)^2\right)\right]\Big|_{t=0} \\
& - \frac{d}{dt}\left[\sum_{k=3}^{+\infty}\frac{1}{k!}\mathbf{s}_0^k\left(\psi_g + tf\right)\right]\Bigg|_{t=0} \\
=& - 4\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\Delta\psi_g\right) - 4\mathrm{tr}\left(\partial\overline{\partial}\psi_g\cdot\partial\overline{\partial}\Delta f\right) - 4\Delta\mathrm{tr}\left(\partial\overline{\partial}\psi_g\cdot\partial\overline{\partial}f\right) \\
& - \frac{d}{dt}\left[\sum_{k=3}^{+\infty}\frac{1}{k!}\mathbf{s}_0^k\left(\psi_g + tf\right)\right]\Bigg|_{t=0} .
\end{aligned}
$$

Setting

$$
\hat{\mathbb{L}}_g f = \frac{d}{dt}\left[\sum_{k=3}^{+\infty}\frac{1}{k!}\mathbf{s}_0^k\left(\psi_g + tf\right)\right]\Bigg|_{t=0}
$$

we have the decomposition

$$
\widetilde{\mathbb{L}}_g f = -4\mathrm{tr}\left(\partial\overline{\partial}\psi_g\cdot\partial\overline{\partial}\Delta f\right) - 4\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\Delta\psi_g\right) - 4\Delta\mathrm{tr}\left(\partial\overline{\partial}\psi_g\cdot\partial\overline{\partial}f\right) - \hat{\mathbb{L}}_g f . \tag{1.4}
$$

Using again Lemma 1.1, we can write explicitly the terms of the quadratic part in f of $\mathbf{s}_0\left(\psi_g + f\right) - \mathbf{s}_0\left(\psi_g\right) - \mathbf{s}_0\left(f\right)$ that depend linearly on ψ_g.

$$
\begin{aligned}
Q_g^{[2]} :=& - 4\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\psi_g\cdot\partial\overline{\partial}\Delta f\right) - 2\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^2\cdot\partial\overline{\partial}\Delta\psi_g\right) \\
& - 8\mathrm{tr}\left(\partial\overline{\partial}\psi_g\cdot\partial\overline{\partial}\mathrm{tr}\left(\left(\partial\overline{\partial}f\right)^2\right)\right) - 16\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\mathrm{tr}\left(\partial\overline{\partial}f\cdot\partial\overline{\partial}\psi_g\right)\right) \\
& - 8\Delta\mathrm{tr}\left(\partial\overline{\partial}\psi_g\cdot\left(\partial\overline{\partial}f\right)^2\right) ,
\end{aligned}
$$

$$
\hat{Q}_g^2\left(f\right) := \mathbf{s}_0\left(\psi_g + f\right) - \mathbf{s}_0\left(\psi_g\right) - \mathbf{s}_0\left(f\right) - \widetilde{\mathbb{L}}_g f - Q_g^{[2]}\left(f\right) .
$$

Setting
$$\tilde{Q}_g^2(f) := Q_g^{[2]}(f) + \hat{Q}_g^2(f) \ ,$$
we have the decomposition of the lemma.

□

1.4 cscK Manifolds

In all of this work we will deal with Kähler manifolds with constant scalar curvature, and now summarize some of their most important properties. The following four results can be found in [Sim04].

Theorem 1.2. *Let (M, J, g) be a compact Kähler manifold and let ξ be a holomorphic vector field on M. The following statements are equivalent:*

- *ξ has a zero somewhere on M;*
- *ξ is tangent to the fibers of the Albanese map $M \to H^0\left(M, \Omega^1\right)^* / H_1(M, \mathbb{Z})$;*
- *there exists a function $f : M \to \mathbb{C}$ such that $\xi = \partial^\sharp f$.*

In particular, the set $\mathfrak{h}_0(M)$ of holomorphic vector fields with zeroes is a linear subspace of $\mathfrak{h}(M) := H^0(M, TM)$and the dimension of the space of holomorphic vector fields of the form $\partial^\sharp f$ is the same for all Kähler metrics g on (M, J).

This result has the following corollary.

Proposition 1.4. *Let (M, J, g) be a compact Kähler manifold. If $f \in C^\infty(M)$ is a real solution of*
$$\left(\overline{\partial}\partial^\sharp\right)^* \overline{\partial}\partial^\sharp f = 0 \,,$$
then im $\left(\partial^\sharp f\right)$ is a Killing field of g. Moreover, a Killing field arises this way iff it has a zero.

Now we state a deep result on the structure of the Lie algebra of holomorphic vector fields of a cscK manifold.

Theorem 1.3. *[Matsushima - Lichnerowicz] Let (M, J, g) a cscK manifold, then the Lie algebra $\mathfrak{h}(M)$ of holomorphic vector fields decomposes as a direct sum*
$$\mathfrak{h}(M) = \mathfrak{h}_0(M) \oplus \mathfrak{a}(M) \ ,$$
where $\mathfrak{a}(M) \subset \mathfrak{h}(M)$ is the abelian subalgebra of parallel holomorphic vector fields and $\mathfrak{h}_0(M)$, letting $\mathfrak{i}(M, g)$ denote the Lie algebra of real Killing vector fields on (M, g), is
$$\mathfrak{h}_0(M) = \left(\mathfrak{i}(M, g) / \mathfrak{a}(M)\right)^{\mathbb{C}} \ .$$
Thus the identity component $Iso_0(M, g)$ of the isometry group is the maximal compact subgroup of the identity component $Aut_0(M, J)$ of the biholomorphism group; $Aut_0(M, J)$ has a compact real form which is a subgroup of $Iso_0(M, g)$; and $Aut_0(M, J)$ is a reductive Lie group.

Remark 1.1. If the Manifold is cscK the fourth order operator
$$\left(\overline{\partial}\partial^\sharp\right)^* \overline{\partial}\partial^\sharp : C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C}) \ ,$$

is actually a real operator

$$\left(\overline{\partial}\partial^{\sharp}\right)^{*}\overline{\partial}\partial^{\sharp}: C^{\infty}(M,\mathbb{R}) \to C^{\infty}(M,\mathbb{R}) \ ,$$

indeed

$$\begin{aligned}
\left(\overline{\partial}\partial^{\sharp}\right)^{*}\overline{\partial}\partial^{\sharp} f &= \nabla_i \nabla^{\bar{j}} \nabla_{\bar{j}} \nabla^i f \\
&= \nabla_i \nabla_j \nabla_{\bar{j}} \nabla_{\bar{i}} f \\
&= \nabla_i \nabla_j \nabla_{\bar{i}} \nabla_{\bar{j}} f \\
&= \nabla_i \nabla_{\bar{i}} \nabla_j \nabla_{\bar{j}} f - \nabla_i \left(R^{\bar{k}}_{j\bar{i}\bar{j}} \nabla_{\bar{k}} f \right) \\
&= \frac{1}{4}\Delta_g^2 f + \nabla_i \left(\mathrm{Ric}_{k\bar{i}} \nabla_{\bar{k}} f \right) \\
&= \frac{1}{4}\Delta_g^2 f + \left(\nabla_i \mathrm{Ric}_{k\bar{i}} \right) \nabla_{\bar{k}} f + \mathrm{Ric}_{k\bar{i}} \nabla_i \nabla_{\bar{k}} f \\
&= \frac{1}{4}\Delta_g^2 f + \frac{1}{2} \left(\nabla_k s_g \right) \nabla_{\bar{k}} f + \mathrm{Ric}_{k\bar{i}} \nabla_i \nabla_{\bar{k}} f \\
&= \frac{1}{4}\Delta_g^2 f + \left\langle \mathrm{Ric}_g, i\partial\overline{\partial} f \right\rangle_g \\
&= \frac{1}{2}\mathbb{L}_g f \, .
\end{aligned}$$

This observation give us a characterization of $\ker(\mathbb{L}_g)$.

Proposition 1.5. *Let (M, g) be a compact cscK manifold, then*

$$\ker(\mathbb{L}_g) = \ker\left(\overline{\partial}\partial^{\sharp}\right)$$

and

$$\dim_{\mathbb{R}}(\ker(\mathbb{L}_g)) = \dim_{\mathbb{C}}(\mathfrak{h}_0(M)) + 1 \, .$$

Proof. Since (M, g) is cscK we have that

$$\mathbb{L}_g = \left(\overline{\partial}\partial^{\sharp}\right)^{*}\overline{\partial}\partial^{\sharp} \, .$$

Let $f \in \ker(\mathbb{L}_g)$ a non constant real function, we have that $\partial^{\sharp} f \in \mathfrak{h}_0(M)$ and by Proposition 1.4

$$im\left(\partial^{\sharp} f\right) \in \mathfrak{i}(M, g) \ ,$$

but it vanishes somewhere so

$$im\left(\partial^{\sharp} f\right) \in \mathfrak{i}(M, g) / \mathfrak{a}(M)$$

and applying Theorem 1.3 we have the thesis. □

Now we focus on the local properties of cscK metrics.

Proposition 1.6. *Let (M, g) be a Kähler manifold with constant scalar curvature, then $g \in C^{\omega}\left(\Omega^{(1,0)}M \otimes \Omega^{(0,1)}M\right)$, that is, the metric g is a real analytic on M.*

Proof. Easy consequence of Proposition 1.3.

□

Remark 1.2. By proposition above, we have that in Kähler normal coordinates at a point $p \in M$ we can decompose the non euclidean part ψ_g of the Kähler potential of g in this way

$$\psi_g(z) = \sum_{k=0}^{+\infty} p_{4+k}\left(z\right) \,.$$

Imposing the equation that satisfies and "filtrating" by degree, we can evince differential relations between the various pieces p_{4+k}. For example

$$-\frac{\Delta^2 p_4}{2} = s_g \tag{1.5}$$

$$-\frac{\Delta^2 p_5}{2} = 0\,. \tag{1.6}$$

This observation yields a lemma.

Lemma 1.4. *Let (M, g) a cscK manifold, for each $p \in M$ in Kähler normal coordinates on $B_{r_0}(p)$ we have that*

$$\omega_g = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + \psi_g\left(z\right)\right) \,,$$

with $\psi_g \in C^\omega\left(B_{r_0}\left(p\right)\right)$

$$\psi_g(z) = \sum_{k=0}^{+\infty} p_{4+k}\left(z\right)$$

and

$$p_4\left(z\right) = -\frac{1}{4} R_{i\bar{j}k\bar{l}}\left(p\right) z^i \overline{z^j} z^k \overline{z^l} = -\frac{s_g|z|^4}{16m(m+1)} + |z|^4\tilde{\phi}_2 + |z|^4\tilde{\phi}_4 \,, \tag{1.7}$$

with $\tilde{\phi}_2, \tilde{\phi}_4$ eigenfunctions of $\Delta_{S^{2m-1}}$ respectively of eigenvalues $-4m, -8(m+1)$.

Proof. By what we have done so far, we only need to prove the last sentence. Since p_4 is a real polynomial in m complex variables, it can be written in terms of real harmonic polynomials in this way

$$p_4 = c_0|z|^4 + |z|^2 H_2\left(z, \overline{z}\right) + H_4\left(z, \overline{z}\right) \,,$$

with $H_k\left(z, \overline{z}\right)$ a harmonic polynomial of degree k, that is a polynomial of degree k such that

$$\Delta H_k = 0\,.$$

Moreover eigenfunctions of $\Delta_{S^{2m-1}}$ come from harmonic polynomials, indeed

$$H_k = |z|^k \phi_k \,,$$

with ϕ_k eigenfunction of $\Delta_{S^{2m-1}}$ of eigenvalue $-k\left(2m-2+k\right)$. Since we know that

$$-\frac{\Delta^2 p_4}{2} = s_g \,.$$

then we have the decomposition

$$p_4 = -\frac{|z|^4 s_g}{16m(m+1)} + |z|^4\tilde{\phi}_2 + |z|^4\tilde{\phi}_4$$

and the lemma is proved. ◻

1.5 ALE Kähler manifolds

Let Γ be a finite subgroup of $SO(m)$ acting freely on S^{m-1} and consider

$$\mathbb{R}^m/\Gamma\,,$$

it is a singular space with only one singular point ('the origin').

Definition 1.4. Let (X_Γ, θ) be a smooth riemannian manifold. We say that it is an *ALE* space with group $\Gamma \lhd SO(m)$ finite and of order $\tau \in \mathbb{R}^+$ if there exist a compact set $K \subset X_\Gamma$ and a map π

$$\pi : X_\Gamma \to \mathbb{R}^m/\Gamma\,,$$

such that

$$\pi : X_\Gamma \setminus K \to \left(\mathbb{R}^m \setminus B_R\right)/\Gamma$$

is a diffeomorphism and the metric θ on $X_\Gamma \setminus B_R$ has an expansion

$$\pi_*\theta = \delta_{ij} + \mathcal{O}\left(|x|^{-\tau}\right)\,.$$

Suppose we have a compact orbifold M of dimension m with, for simplicity, an isolated singular point p of type $\mathbb{R}^m/\Gamma$. If we remove a small ball $B_\varepsilon(p)$ we are left with a smooth manifold with boundary and

$$\partial\left(M \setminus B_\varepsilon(p)\right) \simeq S^{m-1}/\Gamma\,.$$

If we have an ALE space with group Γ we can choose a sufficiently big $R \in \mathbb{R}^+$ such that $\pi^{-1}\left(B_R\right)$ is a smooth manifold with boundary and moreover

$$\partial\pi^{-1}\left(B_R\right) \simeq S^{m-1}/\Gamma\,.$$

Since $M \setminus B_\varepsilon(p)$ and $\pi^{-1}\left(B_R\right)$ have diffeomorphic boundaries we can perform a connected sum construction to get a new smooth manifold $\tilde{M}$ that we can view as a "desingularization" of M. We would like to perform this kind of operation in the Kähler setting. Now let Γ be a finite subgroup of $U(m)$ acting freely on S^{2m-1}.

Definition 1.5. Let (X_Γ, θ) be a Kähler manifold. We say that it is an *ALE* Kähler space with group $\Gamma \lhd U(m)$ finite and of order $\tau \in \mathbb{R}^+$ if there exist a compact set $K \subset X_\Gamma$ and a map

$$\pi : X_\Gamma \to \mathbb{C}^m/\Gamma\,,$$

such that

$$\pi : X_\Gamma \setminus K \to \left(\mathbb{C}^m \setminus B_R\right)/\Gamma$$

is a biholomorphism and the metric θ on $X_\Gamma \setminus B_R\left(0\right)$ has an expansion

$$\pi_*\theta = \frac{\delta_{i\bar{j}}}{2} + \mathcal{O}\left(|x|^{-\tau}\right)\,.$$

Remark 1.3. If (X_Γ, θ) is an ALE Kähler space and X_Γ is an algebraic resolution of the origin, then we have a map

$$\pi : X_\Gamma \setminus \pi^{-1}\left(0\right) \to \mathbb{C}^m/\Gamma \setminus \{0\}\,,$$

that is a biholomorphism and $\pi^{-1}\left(0\right)$ is a compact complex analytic space of complex codimension at least 1.

We are interested in *ALE* Kähler manifolds with metrics that are scalar-flat or Ricci-flat. Kronheimer in [Kro89],[Kro86] classifies in complex dimension 2 Ricci-flat ALE Kähler spaces. Caldebrank - Singer [CS04] construct 2-dimensional scalar-flat ALE Kähler spaces from quotients $\mathbb{C}^2/\mathbb{Z}_k$ for any $U(2)$-action of $\mathbb{Z}_k$ in $\mathbb{C}^2$. Simanca in [Sim91] constructs ALE kahler metrics on the blow up at the origin of $\mathbb{C}^m$. Joyce in [Joy00] proves the following theorem.

Theorem 1.4. *[Joy00, Theorem 8.2.3] Let $m \geq 3$, $\Gamma \triangleleft SU(m)$ finite. If (X, ω) is a crepant resolution of $\mathbb{C}^m/\Gamma$ that is Kähler, then in the same cohomology class of ω there exist an ALE Kähler metric ω' such that*

$$Ric\,(\omega') = 0\,,$$

and outside a compact set ω' has an expansion

$$\omega' = i\partial\overline{\partial}\left(\frac{|x|^2}{2} + c_\Gamma |x|^{2-2m} + \mathcal{O}\left(|x|^{2-2m-\gamma}\right)\right) \qquad \gamma \in (0,1)\,.$$

A special case of the theorem of Joyce is Calabi example: the crepant resolution $\mathbb{C}^m/\mathbb{Z}_m$ with $\mathbb{Z}_m$ acting on $\mathbb{C}^m$

$$(x_1, \ldots, x_m) \to (\zeta_m x_1, \ldots, \zeta_m x_m)$$

with ζ_m a primitive m-th root of unity. Generalizing the construction of Simanca, Rollin and Singer in [RS09b] show that bundles $\mathcal{O}\,(-k)$ can be equipped with scalar-flat metrics. We give here another proof of this fact.

Proposition 1.7. *The total spaces of line bundles $\mathcal{O}_{\mathbb{P}^m}\,(-k)$ are ALE kahler spaces with group $\mathbb{Z}_k$ and of order*

- $2 - 2m$ *if* $k \neq m$
- $-2m$ *if* $k = m$

Proof. The bundle $\mathcal{O}_{\mathbb{P}^m}\,(-1)$ carries a natural bundle metric h coming from its embedding ι in the trivial bundle

$$\left(\mathcal{O}_{\mathbb{P}^m}\,(-1)\,, h\right) \overset{\iota}{\hookrightarrow} \left(\mathbb{P}^m \times \mathbb{C}^{m+1}, h_0\right)$$

with h_0 the standard euclidean metric on $\mathbb{C}^{m+1}$, so $h_1 = \iota^* h_0$. Let $[z_0 : \ldots : z_m]$ be homogeneous coordinates on $\mathbb{P}^m$ and U_i the standard coordinate patch on which $z_i = 1$ we define the coordinates on U_i

$$w_j = \frac{z_j}{z_i} \qquad j \neq i\,.$$

Without loss of generality we can do all the computations in U_0 since all will be equivariant wrt change of charts. The U_i are trivializing charts for $\mathcal{O}_{\mathbb{P}^m}\,(-1)$ and so a generic $v \in \mathcal{O}_{\mathbb{P}^m}\,(-1)$ can be written in a suitable U_i-trivialization as

$$v = (w_1, \ldots, \widehat{w_i}, \ldots, w_m, \lambda) \qquad \lambda \in \mathbb{C}$$

and moreover its norm wrt h_1 can be written as

$$|v|^2_{h_1} = \left(1 + |\mathbf{w}|^2\right)|\lambda|^2\,.$$

Since

$$\mathcal{O}_{\mathbb{P}^m}\,(-k) = \mathcal{O}_{\mathbb{P}^m}\,(-1)^{\otimes k}$$

there is a natural bundle metric h_k coming from h_1 that in coordinate charts can be written

$$|v|^2_{h_k} = \left(1 + |\mathbf{w}|^2\right)^k |\lambda|^2 \,.$$

We look now for a scalar flat ALE metric η_k on $\mathcal{O}_{\mathbb{P}^m}(-k)$ of the following form

$$\eta_k = \partial\overline{\partial}\left(\log\left(|v|^2_{h_k}\right) + \phi\left(|v|^2_{h_k}\right)\right) \,,$$

with $\phi \in C^4([0,+\infty))$ to be determined. By symmetries of the problem we'll extract an ODE from the condition of 0 scalar curvature and its solution will be ϕ. From now on we'll do all calculations on U_0. For the sake of notation we set

$$\begin{aligned} t &= |v|^2_{h_k} \,, \\ x &= \left(1 + |\mathbf{w}|^2\right) \,, \\ t &= x^k |\lambda|^2 \,. \end{aligned}$$

Now we calculate η_k

$$\eta_k = \left[\frac{1}{t} + \phi'(t)\right]\partial\overline{\partial}t + \left[\phi''(t) - \frac{1}{t^2}\right]\partial t \wedge \overline{\partial} t \,.$$

Making the substitution

$$\alpha = 1 + t\phi'(t) \,,$$

we have:

$$\begin{aligned} \partial\overline{\partial}(\log(t) + \phi(t)) = k\alpha\partial\overline{\partial}\log(x) + k^2 t\alpha' \frac{\partial x \wedge \overline{\partial} x}{x^2} + \alpha' x^k d\lambda \wedge \overline{d\lambda} \\ + k\alpha' x^{k-1}\overline{\lambda} d\lambda \wedge \overline{\partial} x + k\alpha' x^{k-1}\lambda \partial x \wedge \overline{d\lambda} \,. \end{aligned}$$

In matrix form η_k is the matrix M

$$M = \begin{pmatrix} \frac{k\alpha}{x}\delta_{ij} + k\frac{\left(kt\alpha' - \alpha\right)}{x^2}\overline{w^i}w^j & k\alpha' x^{k-1}\overline{w^i}\lambda \\ k\alpha' x^{k-1}\overline{\lambda} w^j & \alpha' x^k \end{pmatrix} .$$

To make the notation simpler we define

$$\begin{aligned} A = A_{ij} &= \frac{k\alpha}{x}\delta_{ij} + k\frac{\left(kt\alpha' - \alpha\right)}{x^2}\overline{w^i}w^j \,, \\ W = W_{ij} &= \overline{w^i}w^j \,, \end{aligned}$$

and so

$$A = \frac{k\alpha}{x}\left(I + \frac{\left(kt\alpha^{'} - \alpha\right)}{\alpha x}W\right) ,$$

$$M = \begin{pmatrix} A_{ij} & k\alpha^{'}x^{k-1}\overline{w^i}\lambda \\ k\alpha^{'}x^{k-1}\overline{\lambda}w^j & \alpha^{'}x^k \end{pmatrix} .$$

Now we need to evaluate $\mathrm{tr}_{\eta_k}\left(\mathrm{Ric}\left(\eta_k\right)\right)$ and since

$$\mathrm{tr}_{\eta_k}\left(\mathrm{Ric}\left(\eta_k\right)\right) = \mathrm{tr}\left[M^{-1}\partial\overline{\partial}\log\left(\det\left(M\right)\right)\right] ,$$

we have to compute M^{-1} and $\det(M)$. We omit the long and not illuminating calculations needed to write explicitly M^{-1} and $\det(M)$ since there is only an iterated use of matrix identities that can be found in the Appendix B.

$$\det\left(M\right) = \frac{k^m\alpha^{'}\alpha^m}{x^{m+1-k}} ,$$

$$M^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & \frac{1}{\alpha^{'}x^k} \end{pmatrix} - \begin{pmatrix} 0 & \left(\frac{x}{\alpha}\right)\overline{w^i}\lambda \\ \left(\frac{x}{\alpha}\right)\overline{\lambda}w^k & 0 \end{pmatrix} + \begin{pmatrix} \left(\frac{x^2t\alpha^{'}}{\left(\alpha+|\mathbf{w}|^2kt\alpha^{'}\right)\alpha}\right)W & 0 \\ 0 & \frac{kt|\mathbf{w}|^2}{\alpha x^k} \end{pmatrix} .$$

We set

$$\beta = \log\left(\alpha^{'}\alpha^m\right) ,$$
$$\gamma = \left(k\beta^{'}t - (m+1-k)\right) ,$$

and we get

$$\begin{aligned} \partial\overline{\partial}\log\left(\det\left(M\right)\right) &= \partial\overline{\partial}\log\left(\frac{k^m\alpha^{'}\alpha^m}{x^{m+1-k}}\right) \\ &= \gamma\left[\frac{\partial\overline{\partial}x}{x} - \frac{\partial x\wedge\overline{\partial}x}{x^2}\right] + kt\gamma^{'}\frac{\partial x\wedge\overline{\partial}x}{x^2} + \frac{1}{k}\gamma^{'}x^k d\lambda\wedge\overline{d\lambda} \\ &+ \gamma^{'}x^{k-1}\overline{\lambda}d\lambda\wedge\overline{\partial}x + \gamma^{'}x^{k-1}\lambda\partial x\wedge\overline{d\lambda} . \end{aligned}$$

We set

$$N = \begin{pmatrix} \frac{\gamma}{x}\left[I + \frac{\left(kt\gamma^{'}-\gamma\right)}{x\gamma}W\right] & \gamma^{'}x^{k-1}\lambda\overline{w^i} \\ \gamma^{'}x^{k-1}\overline{\lambda}w^j & \frac{1}{k}\gamma^{'}x^k \end{pmatrix} ,$$

and we finally compute

$$\mathrm{tr}_{\eta_k}\left[\mathrm{Ric}\left(\eta_k\right)\right] = \mathrm{tr}\left(M^{-1}N\right) .$$

Again using simple matrix identities we have

$$\operatorname{tr}\left(M^{-1}N\right) = m\frac{\gamma}{k\alpha} + \frac{\gamma^{'}}{k\alpha^{'}}\,.$$

We now have an ODE of fourth order in ϕ.

$$m\frac{\gamma}{\alpha} + \frac{\gamma^{'}}{\alpha^{'}} = 0\,.$$

Integrating from 0 to t

$$\int_0^t \frac{\gamma^{'}}{\gamma}\,ds = -m\int_0^t \frac{\alpha^{'}}{\alpha} \Rightarrow \log\left(\frac{\gamma}{\gamma_0}\right) = -m\log\left(\frac{\alpha}{\alpha_0}\right)\,ds\,,$$

and using the facts

$$\alpha_0 = 1\,,$$
$$\gamma_0 = kt\beta_0^{'} - (m+1-k) = -\left(m+1-k\right)\,,$$

we get

$$\gamma = -\frac{(m+1-k)}{\alpha^m}\,.$$

Making the substitution

$$\mu = \frac{\alpha^{m+1}}{m+1}\,,$$

the equation becomes

$$kt\frac{\mu^{''}}{\mu^{'}} - (m+1-k) = -\frac{(m+1-k)}{(m+1)^{\frac{m}{m+1}}}\frac{1}{\mu^{\frac{m}{m+1}}}\,.$$

Multiplying by μ' and integrating we obtain the equation

$$kt\alpha^{'}\alpha^m = \alpha^{m+1} - (m+1-k)\,\alpha + (m-k)\,.$$

Recalling that

$$\alpha = 1 + t\phi'\,,$$

we have

$$kt\left(t\phi^{''} + \phi^{'}\right)\left(1+t\phi^{'}\right)^m = \left(1+t\phi^{'}\right)^{m+1} - (m+1-k)\left(1+t\phi^{'}\right) + (m-k)\,.$$

Then setting

$$\phi' = x\,,$$

our equation becomes

$$x^{'} = \frac{\sum_{l=2}^{m+1}\left[\binom{m+1}{l} - k\binom{m}{l-1}\right]t^{l-2}x^l}{k\left(1+tx\right)^m}\,,$$

and hence

$$x^{'} = f(x,t)\,,$$

with f is smooth in a neighborhood of $t=0$, so by existence and uniqueness theorem for ODE, for every initial condition x_0 we have a solution $x \in C^{\infty}\left(\left(-\tau_{x_0}, T_{x_0}\right)\right)$. Let $x_0>0$, we have

$$\alpha = 1+tx \qquad \alpha_0 = 1\,,$$
$$\alpha' = tx^{'} + x \qquad \alpha'(0) = x_0 > 0\,.$$

The polynomial
$$p_{m,k}(x) = x^{m+1} - (m-k+1)\,x + (m-k)$$
has $x=1$ as a root, moreover for $x \geq 1$

$$p'_{m,k}(x) = (m+1)\,x^m - (m+1-k) > 0$$

so we have immediately that $\alpha \in C^{\infty}([0+\infty))$ (and so x) and

$$\alpha, \alpha' > 0\,.$$

We now analyze the asymptotic behavior of α

$$\frac{\alpha'\alpha^m}{\alpha^{m+1} - (m-k+1)\,\alpha + (m-k)} = \frac{1}{kt}\,.$$

Since $\alpha' > 0$, looking at the equation we can't have

$$\lim_{t\to+\infty} \alpha = C \qquad C \in \mathbb{R}^+\,,$$

so we must have
$$\lim_{t\to+\infty} \alpha = +\infty\,.$$

Now it is immediate to recover the asymptotic expansions for α

- for $k \neq m+1$
$$\alpha = c_1 t^{\frac{1}{k}} + c_2 t^{\frac{1-m}{k}} + o\left(t^{-\frac{m}{k}}\right)\,,$$
- for $k = m$
$$\alpha = c_1 t^{\frac{1}{k}} + c_2 t^{\frac{-m}{k}} + o\left(t^{-\frac{(m+1)}{k}}\right)\,.$$

$\square$

In his book [Joy00] Joyce gives an estimate of the potential "at infinity" of a Ricci - flat *ALE* Kähler space coming from a crepant resolution. We will improve that estimate.

Proposition 1.8. *Let (X_Γ, η) be a Ricci flat ALE crepant resolution of an isolated quotient singularity and let $\pi : X_\Gamma \to \mathbb{C}^m/\Gamma$ be the quotient map. Then for $R>0$ large enough, we have that on $X_\Gamma \setminus \pi^{-1}(B_R)$ the Kähler form can be written as*

$$\omega_\eta = i\partial\overline{\partial}\left(\frac{|x|^2}{2} - c_\Gamma |x|^{2-2m} + \psi_\eta(x)\right)\,,$$

where the function ψ_η satisfies the estimate

$$\psi_\eta(x) = \mathcal{O}\left(|x|^{-2m}\right)\,. \tag{1.8}$$

Moreover, the radial component $\psi_\eta^{(0)}$ in the Fourier decomposition of ψ_η is such that

$$\psi_\eta^{(0)}(|x|) = \mathcal{O}\left(|x|^{2-4m}\right)\,.$$

Proof. By [Joy00, Theorem 8.2.3], we have that on $X_\Gamma \setminus \pi^{-1}(B_R)$ the Kähler form ω_η can be written as

$$\omega_\eta \;=\; i\partial\overline{\partial}\left(\frac{|x|^2}{2} - c_\Gamma |x|^{2-2m} + \psi_\eta\,(x)\right) \qquad \text{with} \qquad \psi_\eta\,(x) = \mathcal{O}\left(|x|^{2-2m-\gamma}\right)\,,$$

for some $\gamma \in (0,1)$. Since X_Γ is Ricci flat it is also scalar flat and so by Proposition 1.6 we have that ψ_η is a real analytic function. To obtain the desired estimates on the decay of ψ_η, we are going to make use of the equation $\mathbf{s}_\eta = 0$. By means of identity (1.2) in Lemma 1.1, this can be rephrased in terms of ψ_η as follows

$$\begin{aligned}\frac{1}{2}\Delta^2\psi_\eta \;&=\; 4\mathrm{tr}\left[\partial\overline{\partial}\left(\psi_\eta - c_\Gamma|x|^{2-2m}\right)\cdot\partial\overline{\partial}\Delta\psi_\eta\right] + 2\Delta\mathrm{tr}\left[\left(\partial\overline{\partial}\left(\psi_\eta - c_\Gamma|x|^{2-2m}\right)\right)^2\right]\\ &\quad + Q_0^3\left[\psi_\eta - c_\Gamma|x|^{2-2m}\right]\,,\end{aligned}$$

where, in writing the first summand on the right hand side, we have used the fact that $\Delta|x|^{2-2m} = 0$. Since $\psi_\eta = \mathcal{O}(|x|^{2-2m-\gamma})$, for some $\gamma \in (0,1)$, it is straightforward to see that all of the terms on the right hand side can be estimated as $\mathcal{O}(|x|^{-2-4m-\gamma})$ with the only exception of the purely radial term

$$\Delta\,\mathrm{tr}\big((\partial\overline{\partial}|x|^{2-2m})^2\big) \;=\; \mathcal{O}(|x|^{-2-4m})\,.$$

For sake of convenience, we set now the right hand side of the above equation equal to $F/2$, so that

$$\Delta^2\psi_\eta \;=\; F\,.$$

It is now convenient to expand both ψ_η and F in Fourier series. Let $\left\{\bar{\phi}_{k,1},\ldots,\bar{\phi}_{k,N_k}\right\}$ be a $L^2\left(S^{2m-1}\right)$-orthonormal basis of the k-th eigenspace of $\Delta_{S^{2m-1}}$. Then we have

$$\psi_\eta(x) \;=\; \sum_{k=0}^{+\infty}\sum_{l_k=1}^{N_k}\left(\psi_\eta\right)^{(k)}_{l_k}(|x|)\,\bar{\phi}_{k,l_k}\left(\frac{x}{|x|}\right) \qquad F(x) \;=\; \sum_{k=0}^{+\infty}\sum_{l_k=1}^{N_k}F^{(k)}_{l_k}(|x|)\,\bar{\phi}_{k,l_k}\left(\frac{x}{|x|}\right)\,,$$

Since $\phi_0 \equiv 1$, we will refer to $\psi_\eta^{(0)}$ and $(F)^{(0)}_{l_k}$ as the radial part of ψ_η and F, respectively. We also notice that in the forthcoming discussion it will be important to select among the eigenfunctions $\bar{\phi}_{k,l_k}$'s, only the ones which are Γ-invariant, in order to respect the quotient structure. So far, we have seen that $F^{(0)} = \mathcal{O}(|x|^{-2-4m})$ and $F^{(k)} = \mathcal{O}(|x|^{-2-4m-\gamma})$, for $k \geq 1$. On the other hand, using the linear ODE satisfied by the components $(\psi_\eta)^{(k)}_{l_k}$, it is not hard to see that their general expression is given by

$$(\psi_\eta)^{(k)}_{l_k}(|x|) \;=\; a_{k,l_k}|x|^{4-2m-k} + b_{k,l_k}|x|^{2-2m-k} + c_{k,l_k}|x|^k + d_{k,l_k}|x|^{k+2} + \tilde{\psi}^{(k)}_\eta(|x|)\,,$$

where, in view of the behavior of the $F^{(k)}_{l_k}$'s, the functions $\left(\tilde{\psi}_\eta\right)^{(k)}_{l_k}$ are such that

$$\tilde{\psi}^{(0)}_\eta = \mathcal{O}(|x|^{2-4m}) \qquad \text{and} \qquad \left(\tilde{\psi}_\eta\right)^{(k)}_{l_k} = \mathcal{O}(|x|^{2-4m-\gamma}), \quad \text{for } k \geq 1\,,$$

Since the cited result by Joyce implies that $(\psi_\eta)^{(k)}_{l_k} = \mathcal{O}(|x|^{2-2m-\gamma})$, it is easy to deduce that $c_{k,l_k} = 0 = d_{k,l_k}$, for every $k \in \mathbb{N}$. Moreover, we have that $a_0 = 0 = b_0$ and thus $\psi^{(0)}_\eta = \mathcal{O}(|x|^{2-4m})$, as wanted. The same kind of considerations imply that the components $(\psi_\eta)^{(k)}_{l_k}$'s satisfy the desired estimates for every $k \geq 2$. For $k = 1$, we have that $a_{1,l_1} = 0$, but a priori nothing can be said about the b_{1,l_1}'s and thus at a first glance, one has that

$$(\psi_\eta)^{(1)}_{l_1}(|x|) \;=\; b_{1,l_1}|x|^{1-2m} + \left(\tilde{\psi}_\eta\right)^{(1)}_{l_1}(|x|).$$

Fortunately, it turns out that there are no Γ-invariant eigenfunctions for $\Delta_{S^{2m-1}}$ in the first eigenspace. To see this, we recall that an eigenfunction ϕ_1, is restriction to the unit sphere of a linear function on $\mathbb{C}^m$. If there were a Γ-invariant ϕ_1, then given its associated linear function $\Phi^{(1)}$ and any element $U \in \Gamma$ with $U = \left(U^k_j\right)_{1\leq j,k\leq m}$ then we would have

$$\Phi^{(1)}(w, \overline{w}) = \Phi^{(1)}(Uw, \overline{Uw})$$

and thus

$$\Phi^{(1)}_j = \Phi^{(1)}_i U^i_j\,.$$

In other words, U should have 1 as an eigenvalue with eigenvector $(\Phi^1_1, \ldots, \Phi^1_m)$. This would imply that the action of Γ on S^{2m-1} is not free, which is a contradiction. This means that the components $(\psi_\eta)^{(1)}_{l_1}$, do not appear in the Fourier expansion of ψ_η and hence $\psi_\eta(x) = \mathcal{O}(|x|^{-2m})$. □

Remark 1.4. We want to point out that in the sequel we will use quite often the fact that there are no Γ-invariant eigenfunctions of $\Delta_{S^{2m-1}}$ relative to eigenvalue $2m-1$.

1.6 Gluing constructions

1.6.1 Arezzo - Pacard gluing construction

In [AP06],[AP09], Arezzo and Pacard develope a gluing technique to construct, starting from a cscK manifold, new cscK manifolds that are birational to the original one. We now explain the main ideas behind this Kählerian connected sum construction. In [AP06] they build the theory when the starting manifold is a compact cscK orbifold (M, g) with isolated $U(m)$-singular points and with non degenerate Futaki invariant. Let $p \in M$ and suppose for the moment that M is smooth and has no holomorphic vector fields vanishing somewhere. If we perform the "algebro-geometric" blow up of M at p we get the same topological manifold as if we excised a small ball $B_r(p)$ and we replaced it gluing along the boundary a neighborhood of the origin of the blow up of $\mathbb{C}^m$ at the origin. In this naive gluing procedure we completely forgot of the complex/Kähler structure of the two pieces M and $Bl_0\mathbb{C}^m$. We recall that [Sim91] showed that $Bl_0\mathbb{C}^m$ can be equipped with a ALE Kähler metric η that is moreover scalar flat, so on $\mathbb{C}^m \setminus B_R$

$$\omega_\eta = i\partial\overline{\partial}\left(\frac{|x|^2}{2} + \mathcal{O}\left(|x|^{4-2m}\right)\right)\,.$$

If we take Kähler normal coordinates at $p \in M$ we have that the metric has the form

$$\omega_g = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + \mathcal{O}\left(|z|^4\right)\right)\,,$$

it is, indeed, a small perturbation of the euclidean metric in a small neighborhood of p and exactly euclidean at p. So if we look at the metric g near p and η at "infinity" they are almost the same: the euclidean metric. With these assumptions we can modify the naive gluing procedure in such a way we can construct a Kähler manifold, indeed we can construct a cscK manifold. First take Kähler normal coordinates at p and a small ball $B_{2r}(p)$, then we take a "big" R and we consider $\pi^{-1}(B_R) \subset Bl_0\mathbb{C}^m$ (π is the canonical holomorphic projection $\pi : Bl_0\mathbb{C}^m \to \mathbb{C}^m$). We consider the annulus

$$A_{2r}(p) := B_{2r}(p) \setminus B_r(p)\,,$$

and we rescale the coordinates on it such that it becomes the annulus

$$A_2 := B_2 \setminus B_1\,,$$

and the metric g becomes

$$\omega_g = r^2 i\partial\overline{\partial}\left(\frac{|w|^2}{2} + \mathcal{O}\left(r^2|w|^4\right)\right)\,.$$

Now we do almost the same thing on $Bl_0\mathbb{C}^m$, that is, we take the annulus

$$A_R(p) := \pi^{-1}(B_R) \setminus \pi^{-1}\left(B_{R/2}\right)\,,$$

and we rescale the coordinates on it such that it becomes the annulus

$$A_1 := B_1 \setminus B_{1/2}\,,$$

and the metric η becomes

$$\omega_\eta = R^2 i\partial\overline{\partial}\left(\frac{|w|^2}{2} + \mathcal{O}\left(R^{2-2m}|w|^{4-2m}\right)\right)\,.$$

With a homothety we rescale the metric η to $\frac{r^2}{R^2}\eta$ so on the boundary ∂B_1 the euclidean parts of the two metrics g and $\frac{r^2}{R^2}\eta$ match perfectly. The metrics g and $\frac{r^2}{R^2}\eta$ are very similar at $\partial B_r(p)$ and $\partial\pi^{-1}(B_R)$ but not equal, we have only a "first order" match, but this can be helped. Sice we want to glue $M \setminus B_r(p)$ with $\pi^{-1}(B_R)$ along their boundary and on $B_{2r}(p)$ and $Bl_0\mathbb{C}^m \setminus \pi^{-1}\left(B_{R/2}\right)$we can always write a Kähler metric as a $i\partial\overline{\partial}$ of a function, the problem of gluing continuously two smooth metrics along a sphere translates to the problem of gluing two smooth functions up to derivatives of order 2. In light of this, what we can do is perturb g and $\frac{r^2}{R^2}\eta$ with $i\partial\overline{\partial}$ of functions $F^o \in C^4(M \setminus B_r(p))$,$F^I \in C^4(Bl_0\mathbb{C}^m \setminus \pi^{-1}(B_R))$ that must satisfy the following requests:

- $\omega_g + i\partial\overline{\partial}F^o$ is a cscK metric on $M \setminus B_r(p)$,
- $\frac{r^2}{R^2}\omega_\eta + i\partial\overline{\partial}F^I$ is a cscK metric on $\pi^{-1}(B_R)$,
- potentials of $\omega_g + i\partial\overline{\partial}F^o$ and $\frac{r^2}{R^2}\omega_\eta + i\partial\overline{\partial}F^I$ match up to derivatives of order at least 2 on the common boundary.

Costructing perturbations F^o, F^I is the main issue. Since we can't construct at once F^o, F^I satisfying all the requests we construct two families of functions F^o_h, F^I_k (one on M, one on $Bl_0\mathbb{C}^m$) depending on some parameters h, k and then we will choose right parameters (h, k, r, R) such that the resulting metrics glue. Our F^o_h, F^I_k must be solutions of the equations

$$\mathbf{s}_g\left(F^o_h\right) = \sigma_h \qquad \text{on } M \setminus B_r(p)\,,$$

$$\mathbf{s}_{\frac{r^2}{R^2}\eta}\left(F^I_k\right) = \sigma_h \qquad \text{on } \pi^{-1}(B_R)\,,$$

but we can extend in a suitable way these equations on the complete manifolds, so we can seek for functions F^o_h, F^I_k defined on the whole manifolds. Since we are looking for small perturbations of the reference metrics we can use the decompositions of scalar curvature operator developed in this chapter to set up a fixed point problem to solve in suitable functional spaces. To set up this fixed point problem on M we need to invert the linearization of the scalar curvature operator, and here comes into play the fact that M has no holomorphic vector fields vanishing somewhere, indeed this condition guarantees injectivity and the invertibility of

$$\mathbb{L}_g : W^{4,p}(M)/\mathbb{R} \to L^p(M)/\mathbb{R} \qquad p > 1\,.$$

To set up the fixed point problem on $Bl_0\mathbb{C}^m$ we need to invert $\mathbb{L}_\eta$ too and we can achieve this choosing the right functional spaces (Holder spaces of functions decaying at infinity). Since we want a control on the behavior at the boundary of F^o_h, F^I_k we impose a particular form for F^o_h, F^I_k that is

$$F^o_h = H^o_h + f^o_h\,,$$

$$F^I_k = H^I_k + f^I_k\,,$$

with H^o_h a suitably cut off and rescaled version of euclidean biharmonic extension B^o_h of functions on the sphere $(h_1, h_2) \in C^4(S^{2m-1}) \times C^2(S^{2m-1})$ on $\mathbb{C}^m \setminus B_1$

$$\begin{cases} \Delta^2 B^o_h = 0 & \text{on } \mathbb{C}^m \setminus B_1 \\ B^o_h = h_1 & \text{on } \partial B_1 \\ \Delta B^o_h = h_2 & \text{on } \partial B_1 \end{cases}$$

and H^I_k a suitably cut off and rescaled version of euclidean biharmonic extension B^I_k of functions on the sphere $(k_1, k_2) \in C^4(S^{2m-1}) \times C^2(S^{2m-1})$ on B_1

$$\begin{cases} \Delta^2 B^I_k = 0 & \text{on } B_1 \\ B^I_k = k_1 & \text{on } \partial B_1 \\ \Delta B^I_k = k_2 & \text{on } \partial B_1 \end{cases}.$$

Why this decomposition? The linearizations of the scalar curvature operator $\mathbb{L}_g$ and $\mathbb{L}_\eta$ have the principal parts that are Δ^2_g and Δ^2_η and since the noneuclidean parts of the metrics are decaying they are "almost" the euclidean laplacian, so H^o_h and H^I_k are "almost" in their kernel and let us to stay "near" background metrics. We fix H^o_h and H^I_k and we seek for perturbations f^o_h, f^I_k such that

$$\mathbf{s}_g\left(H^o_h + f^o_h\right) = \sigma_h \qquad \text{on } M \setminus B_r(p)\,,$$

$$\mathbf{s}_{\frac{r^2}{R^2}\eta}\left(H^I_k + f^I_k\right) = \sigma_h \qquad \text{on } \pi^{-1}\left(B_R\right)\,.$$

Working out some delicate estimates Arezzo and Pacard prove the existence of f^o_h f^I_k for every h, k in a suitable subset of $C^4(S^{2m-1}) \times C^2(S^{2m-1})$. Then looking at the behavior of the potentials at the respective boundaries we can set up a second fixed point problem (the so called Cauchy data matching) on $C^4(S^{2m-1}) \times C^2(S^{2m-1})$ and using the estimates on f^o_h, f^I_k, Arezzo and Pacard are able to prove that there exist a choice of h, k such that $\omega_g + i\partial\overline{\partial}F^o$ and $\frac{r^2}{R^2}\omega_\eta + i\partial\overline{\partial}F^I$ glue to a cscK metric on Bl_pM. This is a oversimplified version of their true result that is the following theorem.

Theorem 1.5. *Let (M, g) a cscK orbifold with isolated singularities that is Futaki non degenrate. Let $n \geq 0$, $p_1, \ldots p_n \in M$ with neighborhoods biholomorphic to neighborhoods of the origin of $\mathbb{C}^m/\Gamma_i$ with $\Gamma_i \triangleleft U(m)$ finite (even trivial). Suppose that every $\mathbb{C}^m/\Gamma_i$ admit an ALE Kähler* *resolution* (X_i, η_i). *Then there exist an* $\varepsilon_0 > 0$ *such that* $\forall \varepsilon \in (0, \varepsilon_0)$ *there exist a constant scalar curvature Kähler metric* g_ε *on the space*

$$\tilde{M} = M \sharp_{p_1} X_1 \cdots \sharp_{p_n} X_n \,,$$

that as ε *tends to* 0 *the sequence* g_ε *converges to* g *in* C^∞ *topology away from points* p_i. *If* s_g *is positive or negative so is* s_{g_ε} *Moreover if* M *has no holomorphic vector fields vanishing somewhere*

$$[\omega_{g_\varepsilon}] = \pi^*[\omega_g] + \varepsilon^2 \left(\sum_{k=1}^{n} [\eta_k] \right) \,,$$

with $\pi : \tilde{M} \to M$ *the canonical holomorphic surjection.*

In their second work on the subject [AP09], they start from a cscK manifold (M, g) with holomorphic vector fields vanishing somewhere. This fact tells us that it is not always possible to find a cscK metric on the blown up manifold, indeed for example $Bl_p\mathbb{P}^2$ doesn't admit a cscK metric since its automorphism group isn't reductive. Because of the presence of holomorphic vector fields vanishing somewhere, the operator $\mathbb{L}_g$ has non constant functions in its kernel and so it is not easily invertible any more. To overcome this problem we look more carefully to the ALE structure of $Bl_0\mathbb{C}^m$. At infinity we have the expansion

$$\omega_\eta = i\partial\overline{\partial} \left(\frac{|x|^2}{2} + c_1|x|^{4-2m} + c_2|x|^{2-2m} + \mathcal{O}(|x|^{6-4m}) \right) \,.$$

We note that the function $|w|^{4-2m}$ is in the ker of Δ^2 and so $|w|^{4-2m}$, once suitably cut off to 0 on a compact set containing the exceptional divisor of $Bl_0\mathbb{C}^m$, it is "almost" in $\ker(\mathbb{L}_\eta)$. We fix a point $p \in M$ and we take normal coordinates centered at it, if we bring the function $|x|^{4-2m}$ suitably cut off to $B_{2r}(p)$ more precisely a perturbation of it

$$G_p(z) := \chi \left(|z|^{4-2m} + \mathcal{O}\left(|z|^{6-2m} \right) \right) \,,$$

it blows up at p, but in this case too is "almost" in $\ker(\mathbb{L}_g)$ indeed

$$\mathbb{L}_g G_p = c_m \delta_p + \phi \qquad \phi \in C^\infty(M) \,.$$

We recall that an equation on M of type

$$P(u) = \mu \qquad \mu \in \mathcal{D}'(M)$$

with P elliptic and μ a distribution has a solution if and only if

$$\langle \mu, \psi \rangle = 0 \qquad \forall \psi \in \ker(P^*) \,.$$

So if $f \in L^p(M)$ and we consider equations of type

$$\mathbb{L}_g u = a_0 + \sum_{j=1}^{N} a_j \delta_{p_j} + f \qquad a_j \in \mathbb{R} \,,$$

if points p_j are "enough" and "well disposed" we can find a_j such that there exist a solution u (and actually prove some regularity). So the price we pay for the inversion of $\mathbb{L}_g$ is that we have functions that blow up at points p_k. But we are interested in functions on $M \setminus \left(\bigcup_{j=1}^{n} B_r(p_j) \right)$

so on this space we actually don't have functions that blow up. As a first step we seek for a solution of equation

$$\mathbb{L}_g H_a = a_0 + \sum_{j=1}^{N} a_k \delta_{p_j} \qquad \text{for } j \geq 1 \quad a_j \in \mathbb{R}^+ \,,$$

the existence of a solution to this kind of equation imposes a geometric condition on points p_j that we call "balancing condition". If we set

$$\varepsilon = \frac{r}{R} \,,$$

and if the H_a exists and we look at

$$\omega_{g'} = \omega_g + i\varepsilon^{2m} \partial\overline{\partial} H_a \,,$$

on $M \setminus \left(\bigcup_{j=1}^{n} B_r(p_j) \right)$ and to

$$\eta_j' = c_j(a)^2 \varepsilon^2 \eta_j$$

on X_j for suitably chosen $c_j(a)$ we have a "second order" match for potentials, that is, not only we match the euclidean part of potentials, but we match the $|x|^{4-2m}$ part on the ALE space with the asymptotic $|z|^{4-2m}$ on M_ε coming from $\varepsilon^{2m} H_a$. In light of this observation we look for $F_h^o \in C^4 \left(M \setminus \left(\bigcup_{j=1}^{n} B_r(p_j) \right) \right)$, $F_k^I \in C^4 \left(\pi^{-1} \left(B_{\frac{R}{c_j(a)}} \right) \right)$ solutions of the equations

$$\mathbf{s}_g \left(\varepsilon^{2m} H_a + F_h^o \right) = \sigma_h \qquad \text{on } M \setminus \left(\bigcup_{j=1}^{n} B_\varepsilon(p_j) \right) \,,$$

$$\mathbf{s}_{c_j(a)^2 \varepsilon^2 \eta_j} \left(F_k^I \right) = \sigma_h \qquad \text{on } \pi^{-1} \left(B_{\frac{R}{c_j(a)}} \right) \,.$$

The "second order match" is the key fact that Arezzo and Pacard use to set up, in this new setting, the fixed point problems as in [AP06]. This more refined preparation of metrics g and η_k let them to guarantee, also in this new setting, the existence of F_h^o and F_k^I with nice enough estimates. At this point the very same strategy of [AP06] applies and they are able to prove the following theorem.

Theorem 1.6. *Let (M, g) be a cscK manifold, and let $\varphi_1, \ldots, \varphi_d \in C^\infty(M)$ such that*

$$\ker(\mathbb{L}_g) := \mathit{span}_{\mathbb{R}} \{1, \varphi_1, \ldots, \varphi_d\} \,.$$

Let $p_1, \ldots, p_n \in M$ with $n \geq d+1$ and

$$\Phi = \left(\varphi_i(p_j) \right)_{1 \leq i \leq d, 1 \leq j \leq n} \,.$$

If

$$rk(\Phi) = d$$

and there exist $\boldsymbol{a} := (a_1, \ldots, a_n) \in (\mathbb{R}^+)^n$ satisfying

$$\Phi \boldsymbol{a} = 0$$

then there exist ε_0 *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *there exist a cscK metric* g_ε *on* $Bl_{p_1,\cdots,p_n}M$ *such that*

$$[\omega_{g_\varepsilon}] = \pi^*[\omega_g] - \varepsilon^2 \left(\sum_{k=1}^{n} \hat{a}_k^{\frac{1}{m-1}} \left[c_1\left(\mathcal{O}\left(E_k\right)\right)\right] \right)$$

with E_k *the exceptional divisor at point* p_k *and*

$$\hat{a}_k \to a_k \qquad \text{as } \varepsilon \to 0\,.$$

A question now arises naturally: is the theory of [AP09] extendable in some way to the case of a cscK orbifold with isolated singularities with holomorphic vector fields vanishing somewhere? This is the purpose of the present work.

1.6.2 Gluing construction for orbifold with vector fields

We now give the definition of Kähler orbifold with isolated singularities that is a particular type of complex analytic space, we refer to [Dem12] for the definition of complex analytic spaces.

Definition 1.6. A Kähler orbifold of dimension m with isolated singularities (M, ω_g) is a complex analytic space whose singular set consists of isolated points. Each point in the singular set has a neighborhood biholomorphic to B_r/Γ with Γ a finite subgroup of $U(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$. The orbifold Kähler form ω_g is a smooth Kähler form outside the singular locus and, on a neighborhood $U(p)$ of a singular point p such that $U(p) \setminus \{p\} \simeq B_r \setminus \{0\}/\Gamma$, ω_g lifts to a smooth Kähler form on $B_r \setminus \{0\}$ that extends smoothly through the origin to a smooth Kähler form on B_r.

Remark 1.5. All results stated in section 1.4 hold true for compact cscK orbifolds with isolated singularities.

Suppose we have a compact cscK orbifold with isolated singularities (M, g) of dimension $m \geq 3$ and let

$$\mathbf{p} := \{p \in M | p \text{ is a } SU(m) \text{ singularity admitting a Kähler crepant resolution}\}$$

We want to desingularize these points by gluing a suitable model space and we want to get on the resulting manifold a metric with constant scalar curvature. More precisely we want to prove the following theorem.

Theorem 1.7. *Let* (M, g) *be a compact cscK orbifold with isolated singularities and let*

$$\boldsymbol{p} := \{p \in M | p \textit{ is a } SU(m) \textit{ singularity admitting a Kähler crepant resolution}\}$$

and

$$\ker\left(\mathbb{L}_g\right) = \langle 1, \varphi_1, \ldots, \varphi_d \rangle\ .$$

Suppose moreover that

- $\sharp \boldsymbol{p} = N \geq d + 1$,
- *the* $d \times N$ *matrix*
$$\Delta\Phi\left(\boldsymbol{p}\right)_{i,j} := \Delta_g \varphi_i\left(p_j\right) + s_g \varphi_i\left(p_j\right) \tag{1.9}$$
has full rank,

- *there exist* $\boldsymbol{b} := (b_1, \ldots, b_N) \in \mathbb{R}_+^N$ *such that*

$$\sum_{j=1}^{N} b_j \left[\Delta_g \varphi_i \left(p_j\right) + s_g \varphi_i \left(p_j\right)\right] = 0 \quad 1 \le i \le d\,. \tag{1.10}$$

Then there exist $\left(\tilde{M}, \tilde{g}_{\boldsymbol{b}}\right)$ *cscK orbifold together with a holomorphic, surjective map*

$$\pi : \tilde{M} \to M$$

obtanied replacing $\boldsymbol{p}$ *with ALE-Kähler spaces that are Ricci-flat.*

We now illustrate in some detail the strategy we want to use to perform the gluing construction.

Step 1: We relate each point p_j to a suitable ALE space X_j.

Step 2: We take Kähler normal coordinates $(z_1, \ldots, z_m)$ centered at p_j and we cut from M the balls B_r with r small and we get the smooth manifold with boundary

$$M_r := M \setminus \left(\bigcup_{j=1}^{N} B_r(p_j) \right) .$$

We also define $M_{\mathbf{p}}$ as

$$M_{\mathbf{p}} := M \setminus \mathbf{p}\,.$$

Step 3: We take coordinates at infinity $(x_1, \ldots, x_m)$ for the ALE spaces X_j and we take a "big" compact submanifold with boundary

$$X_{R,j} := \pi_j^{-1}\left(B_R\right) .$$

Step 4: We introduce a small parameter ε and we impose that the quantities r, R respectively the radii of balls excised by M and the inner radii of annuli excised from the ALE spaces satisfy the following relations

$$r = r_\varepsilon := \varepsilon^{1-\lambda}\,,$$
$$R = R_\varepsilon := \varepsilon^{-\lambda}\,,$$

with $\lambda = \frac{2}{2m+1}$.

Step 5: We want to construct a family of cscK metrics on M_{r_ε} that depend on ε and other suitable parameters. This family of metrics must be a perturbation of the base metric by the $i\partial\overline{\partial}$ of a function $\mathbb{F}^o_{\mathbf{b},\mathbf{hk}}$:

$$\omega_{g_{\mathbf{b},\mathbf{hk}}} = \omega_g + i\partial\overline{\partial}\mathbb{F}^o_{\mathbf{b},\mathbf{hk}}\,.$$

We will construct this function in such a way that we can prescribe the behavior of the family of metrics at the truncation loci; to do this we will use outer biharmonic extensions of functions on the sphere that are invariant under the action of some finite subgroups of $SU(m)$. So our function $\mathbb{F}^o_{\mathbf{hk}}$ will have the following form:

$$\mathbb{F}^o_{\mathbf{b},\mathbf{hk}} := \tilde{H}^o_{\mathbf{hk}} + \mathbb{H}^b_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\,.$$

In the following five steps we explain how we construct the three components of $\mathbb{F}^o_{\mathbf{b},\mathbf{hk}}$.

Step 6: We start constructing the "skeleton" of $\mathbb{F}^o_{\mathbf{b},\mathbf{hk}}$: $\mathbb{H}^b_{\mathbf{hk}}$. We will do it in many steps, perturbing more and more a function $H^{\mathbf{b}}$ that we can construct "by hand". Assuming that hypotheses of Theorem 1.7 are satisfied we can find $\mathbf{b} \in (\mathbb{R}^+)^N$ and construct a function $H^{\mathbf{b}}$ on $M_{\mathbf{p}}$ such that

$$\mathbb{L}_g H^{\mathbf{b}} + b_0 = 0\,,$$

and on neighborhoods of points p_j has expansion

$$H^{\mathbf{b}} \approx b_j c_{\Gamma_j}\left(-|z|^{2-2m} + \frac{(m-1)\,s_g}{2\,(m-2)\,m\,(m+1)}|z|^{4-2m}\right).$$

Why this kind of function? The term $-b_j c_{\Gamma_j}|z|^{2-2m}$, once suitably rescaled, has exactly the same asymptotic behavior of the first non euclidean asymptotic of the potential of the metric η_j, so in the sequel we will use this property to have a "second order match" at the boundaries for the two families of metrics we will construct. The term with growth $|z|^{4-2m}$ will also be crucial to have a "second order match" but its relevance will be clear when we will build the families of metrics on model spaces. This step is Lemma 3.1.

Step 7: We operate the first modification to $H^{\mathbf{b}}$. We take any $\mathbf{h}^{(0)}, \mathbf{k}^{(0)} \in \mathbb{R}^N$ such that

$$\left|h_j^{(0)}\right|, \left|k_j^{(0)}\right| \leq \kappa r_\varepsilon^\beta \qquad 0 \leq j \leq N\,,$$

with

$$r_\varepsilon^\beta = \varepsilon^{4m+2} r_\varepsilon^{-4m-\tau} \qquad \tau \in (0,1)$$

and $\kappa > 0$ to be determined. Again if we assume that hypotheses of Theorem 1.7 are satisfied we can find $\tilde{\mathbf{b}} \in (\mathbb{R}^+)^n$ and construct a function $H^b_{\mathbf{hk}}$ such that

$$\mathbb{L}_g H^b_{\mathbf{hk}} + \tilde{b}_0 = 0\,,$$

and on neighborhoods of points p_j has expansion

$$\begin{aligned} H^b_{\mathbf{hk}} \approx & \tilde{b}_j c_{\Gamma_j}\left(-|z|^{2-2m} + \frac{(m-1)\,s_g}{2\,(m-2)\,m\,(m+1)}|z|^{4-2m}\right) \\ & + \left(h^{(0)} + \frac{k^{(0)}}{4(m-2)}\right)\varepsilon^{-2m}\left|\frac{z}{r_\varepsilon}\right|^{2-2m} - \frac{k^{(0)}\varepsilon^{-2m}}{4(m-2)}\left|\frac{z}{r_\varepsilon}\right|^{4-2m}. \end{aligned}$$

We note that the quantities

$$\left(h^{(0)} + \frac{k^{(0)}}{4(m-2)}\right)\frac{r_\varepsilon^{2m-4}}{\varepsilon^{2m}}, \frac{k^{(0)}\varepsilon^{-2m}}{4(m-2)}\frac{r_\varepsilon^{4-2m}}{\varepsilon^{2m}}$$

are small since they are positive powers of ε. The reason for this modification will become clear when we will construct the function $\tilde{H}^o_{\mathbf{hk}}$. This step is Lemma 3.2.

Step 8: We modify $H^b_{\mathbf{hk}}$. Let (X_j, η_j) be the ALE space associated to p_j. We know that outside a compact set we have

$$\omega_{\eta_j} = i\partial\overline{\partial}\left(\frac{|x|^2}{2} - c_{\Gamma_j}|x|^{2-2m} + \psi_{\eta_j}(x)\right).$$

We take ψ_{η_j}, we "bring them" to M rescaling and cutting off them and we add the resulting functions to $H^b_{\mathbf{hk}}$ obtaining our skeleton $\mathbb{H}^b_{\mathbf{hk}}$:

$$\mathbb{H}^b_{\mathbf{hk}} = \varepsilon^{2m} H^b_{\mathbf{hk}} + \sum_{j=1}^{N} \varepsilon^2 \bar{b}_j^2 \tilde{\chi}_{j,r_0} \psi_{\eta_j} \left(\frac{z}{\bar{b}_j \varepsilon} \right)$$

with

$$\bar{b}_j := \sqrt[2m]{\tilde{b}_j}$$

and $\tilde{\chi}_{j,r_0}$ smooth cutoff functions

$$\tilde{\chi}_{j,r_0}(p) : \begin{cases} 0 & p \in B_{r_\varepsilon/3}(p_j) \\ 1 & p \in B_{r_0}(p_j) \setminus B_{\frac{r_\varepsilon}{2}}(p_j) \\ 0 & p \in M \setminus B_{2r_0}(p_j) \end{cases}$$

We add potentials ψ_{η_j} to improve further "the second order match" at the boundaries. This step is explained at the end of subsection 3.1.2. This trick of bringing to M the whole Kähler potential of the metric of the ALE space is inspired to the construction that Szekelyhidi performs in [Szé12].

Step 9: We construct $\tilde{H}^o_{\mathbf{hk}}$. This piece of $\mathbb{F}^o_{\mathbf{b},\mathbf{hk}}$ will prescribe the behavior of the family $g_{\mathbf{b},\mathbf{hk}}$ at $\partial M_{r_\varepsilon}$. More precisely, we want that at $\partial B_{r_\varepsilon}(p_j)$

$$\begin{aligned} \omega_{g_{\mathbf{b},\mathbf{hk}}} \approx & \omega_g + i\partial\overline{\partial}\mathbb{H}^b_{\mathbf{hk}} + i\partial\overline{\partial}\tilde{H}^o_{\mathbf{hk}} \\ = & \omega_g + i\partial\overline{\partial}\left(-\bar{b}_j^{2m} c_{\Gamma_j} \varepsilon^{2m} |z|^{2-2m} + \frac{\bar{b}_j^{2m} c_{\Gamma_j} (m-1) s_g \varepsilon^{2m}}{2(m-2)m(m+1)} |z|^{4-2m} + \varepsilon^2 \bar{b}_j^2 \psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon} \right) \right) \\ & + i\partial\overline{\partial}\left(\left(h^{(0)} + \frac{k^{(0)}}{4(m-2)} \right) |z|^{2-2m} - \frac{k^{(0)}}{4(m-2)} |z|^{4-2m} + \tilde{H}^o_{\mathbf{hk}} \right). \end{aligned}$$

We are asking, indeed, $f^o_{\mathbf{b},\mathbf{hk}}$ to be much "smaller" than $\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}}$. To this aim we construct $\tilde{H}^o_{\mathbf{hk}}$ starting from euclidean outer biharmonic extensions of well chosen functions on the unit sphere. We take

$$\left(\mathbf{h}^{(\dagger)}, \mathbf{k}^{(\dagger)} \right) \in C^{4,\alpha}\left(S^{2m-1}\right)^N \times C^{2,\alpha}\left(S^{2m-1}\right)^N$$

such that

$$\int_{S^{2m-1}} h_j^{(\dagger)} d\mu_0 = \int_{S^{2m-1}} k_j^{(\dagger)} d\mu_0 = 0$$

and

$$\left\| h_j^{(\dagger)} \right\|_{C^{4,\alpha}(S^{2m-1})}, \left\| k_j^{(\dagger)} \right\|_{C^{4,\alpha}(S^{2m-1})} \leq \kappa r_\varepsilon^\sigma$$

with $\kappa > 0$ the same of step 7 to be determined and

$$r_\varepsilon^\sigma = \varepsilon^{2m+4} r_\varepsilon^{-2-2m-\tau} \qquad \tau \in (0,1).$$

Euclidean outer biharmonic extension $H^o_{h_j^{(\dagger)} k_j^{(\dagger)}}$ of $h_j^{(\dagger)}, k_j^{(\dagger)}$ is the solution of the boundary value problem

$$\begin{cases} \Delta^2 H^o_{h_j^{(\dagger)} k_j^{(\dagger)}} = 0 & w \in \mathbb{C}^m \setminus B_1 \\ H^o_{h_j^{(\dagger)} k_j^{(\dagger)}} = h_j^{(\dagger)} & w \in \partial B_1 \\ \Delta H^o_{h_j^{(\dagger)} k_j^{(\dagger)}} = k_j^{(\dagger)} & w \in \partial B_1 \end{cases}$$

and we define $\tilde{H}^o_{\mathbf{hk}}$ "bringing to M" euclidean biharmonic extensions rescaling and cutting off them:

$$\tilde{H}^o_{\mathbf{hk}} := \sum_{j=1}^{N} \chi_{r_0,j} H^o_{h_j^{(\dagger)} k_j^{(\dagger)}} \left(\frac{z}{r_\varepsilon} \right)$$

with $\chi_{r_0,j}$ smooth cutoff functions

$$\chi_{r_0,j}(p) := \begin{cases} 1 & p \in B_{r_0}(p_j) \\ 0 & p \in M \setminus B_{2r_0}(p_j) \end{cases}$$

The function $\tilde{H}^o_{\mathbf{hk}}$ will be the crucial tool for the gluing procedure. When we will have families of cscK metrics on M_{r_ε} and on truncated model spaces, the particular form of this function, will allow us to set up a fixed point problem on $C^{4,\alpha}\left(S^{2m-1}\right)^N \times C^{2,\alpha}\left(S^{2m-1}\right)^N$ to find right parameters $\mathbf{h}^{(0)}, \mathbf{k}^{(0)}, \mathbf{h}^{(\dagger)}, \mathbf{k}^{(\dagger)}$ to conclude the gluing procedure. We put restriction on functions $\mathbf{h}^{(\dagger)}, \mathbf{k}^{(\dagger)}$: their means have to vanish; so they can vary only in a closed set of finite codimension in $C^{4,\alpha}\left(S^{2m-1}\right)^N \times C^{2,\alpha}\left(S^{2m-1}\right)^N$. To set up a fixed point problem and have some sort of compactness we will need to move in a closed set with non empty interior so we have to regain these degree of freedom we lost. Here comes into play $\mathbb{H}^b_{\mathbf{hk}}$ in particular $H^b_{\mathbf{hk}}$ we defined in step 7. Indeed on $B_{r_\varepsilon}(p_j)$

$$\begin{aligned} \mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} \approx & \left(-\bar{b}_j^{2m} c_{\Gamma_j} \varepsilon^{2m} |z|^{2-2m} + \frac{\bar{b}_j^{2m} c_{\Gamma_j} (m-1) s_g \varepsilon^{2m}}{2(m-2)m(m+1)} |z|^{4-2m} \right) \\ & + \varepsilon^2 \bar{b}_j^2 \chi_{r_0,p_j} \psi_{\eta_j} \left(\frac{z}{\bar{b}_j \varepsilon} \right) \\ & + \left(\left(h_j^{(0)} + \frac{k_j^{(0)}}{4(m-2)} \right) |z|^{2-2m} - \frac{k^{(0)}}{4(m-2)} |z|^{4-2m} + H^o_{h_j^{(\dagger)} k_j^{(\dagger)}} \left(\frac{z}{r_\varepsilon} \right) \right) \\ \approx & \left(-\bar{b}_j^{2m} c_{\Gamma_j} \varepsilon^{2m} |z|^{2-2m} + \frac{\bar{b}_j^{2m} c_{\Gamma_j} (m-1) s_g \varepsilon^{2m}}{2(m-2)m(m+1)} |z|^{4-2m} \right) \\ & + \varepsilon^2 \bar{b}_j^2 \chi_{r_0,p_j} \psi_{\eta_j} \left(\frac{z}{\bar{b}_j \varepsilon} \right) \\ & + H^o_{h_j^{(0)} + h_j^{(\dagger)}, k_j^{(0)} + k_j^{(\dagger)}} \left(\frac{z}{r_\varepsilon} \right) \end{aligned}$$

and so, as we see in the last line above, we regain the freedom we lost when we put together $\mathbb{H}^b_{\mathbf{hk}}$ and $\tilde{H}^o_{\mathbf{hk}}$. We see now that our "boundary conditions" will be functions

$$(\mathbf{h}, \mathbf{k}) \in C^{4,\alpha}\left(S^{2m-1}\right)^N \times C^{2,\alpha}\left(S^{2m-1}\right)^N$$

such that their means $h_j^{(0)}, k_j^{(0)}$

$$h_j^{(0)} := \frac{1}{\mu\left(S^{2m-1}\right)} \int_{S^{2m-1}} h_j d\mu_0 \qquad k_j^{(0)} := \frac{1}{\mu\left(S^{2m-1}\right)} \int_{S^{2m-1}} k_j d\mu_0$$

satisfy the estimate

$$\left|h_j^{(0)}\right|, \left|k_j^{(0)}\right| \leq \kappa r_\varepsilon^\beta$$

and their "non-radial parts" $h_j^{(\dagger)}, k_j^{(\dagger)}$

$$h_j^{(\dagger)} := h_j - h_j^{(0)} \qquad k_j^{(\dagger)} := k_j - k_j^{(0)}$$

satisfy the estimate

$$\left\|h_j^{(\dagger)}\right\|_{C^{4,\alpha}\left(S^{2m-1}\right)}, \left\|k_j^{(\dagger)}\right\|_{C^{2,\alpha}\left(S^{2m-1}\right)} \leq \kappa r_\varepsilon^\sigma$$

For the sake of notation we will define

$$\mathcal{B}_\alpha := \left\{(\mathbf{h}, \mathbf{k}) \in C^{4,\alpha}(\partial B_1)^N \times C^{2,\alpha}(\partial B_1)^N \mid h_j, k_j \text{ are } \Gamma_j - \text{invariant}\right\}$$

and $\mathfrak{B}(\kappa, \beta, \sigma) \subset \mathcal{B}_\alpha$ the set of functions $(\mathbf{h}, \mathbf{k}) \in \mathcal{B}_\alpha$ satisfying conditions above. One should note that this construction is significatively different than the one in [AP09]. Indeed, $\mathbf{h}^{(0)}, \mathbf{k}^{(0)}$ aren't involved in this step but they are included in step 9. This kind of construction is necessary since if we try to imitate the very same construction of [AP09] we won't have correct estimates to perform the data matching. This step is subsection 3.1.1.

Step 10: We construct the last piece of $\mathbb{F}^o_{\mathbf{b},\mathbf{hk}}$: $f^o_{\mathbf{b},\mathbf{hk}}$. This term has to assure the constancy of the scalar curvature of $g_{\mathbf{b},\mathbf{hk}}$ on M_{r_ε}. We construct this function solving a PDE on suitably extended weighted Hölder spaces on M. More precisely we will look for $f^o_{\mathbf{b},\mathbf{hk}}$ in $C^{4,\alpha}_{4-2m+\tau}\left(M_{\mathbf{p}}\right) \oplus \mathcal{D}$ with $\tau \in (0,1)$ and $\mathcal{D}$ a finite dimensional subspace of $C^{4,\alpha}_{2-2m}\left(M_{\mathbf{p}}\right) \cap C^\infty_{loc}\left(M_{\mathbf{p}}\right)$ of functions H_d that on $B_{r_0}\left(p_j\right)$ have expansion

$$H_d \approx d_j c_{\Gamma_j}\left(-|z|^{2-2m} + \frac{(m-1)\, s_g}{2\,(m-2)\, m\,(m+1)}|z|^{4-2m}\right).$$

So we can decompose further $f^o_{\mathbf{b},\mathbf{hk}}$ as

$$f^o_{\mathbf{b},\mathbf{hk}} := \tilde{f}^o_{\mathbf{b},\mathbf{hk}} + H_f$$

with $\tilde{f}^o_{\mathbf{b},\mathbf{hk}} \in C^{4,\alpha}_{4-2m+\tau}\left(M_{\mathbf{p}}\right)$ and $H_f \in \mathcal{D}$ and, on $B_{r_0}\left(p_j\right)$, $f^o_{\mathbf{b},\mathbf{hk}}$ has expansion

$$H_f \approx \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} c_{\Gamma_j}\left(-|z|^{2-2m} + \frac{(m-1)\, s_g}{2\,(m-2)\, m\,(m+1)}|z|^{4-2m}\right)$$

with $\hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \in \mathbb{R}$. We complete the construction of $\mathbb{F}^o_{\mathbf{b},\mathbf{hk}}$ and hence we obtain our family of cscK metrics $g_{\mathbf{b},\mathbf{hk}}$ on M_{r_ε}. This step is developed in section 3.1.3.

Step 11: Once we will have the family $g_{\mathbf{b},\mathbf{hk}}$ we look at the expansion of its potential on $B_{2r_\varepsilon}\left(p_j\right) \setminus B_{r_\varepsilon}\left(p_j\right)$. We recall that on B_{r_0} the metric g can be written as

$$\omega_g = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + \psi_g(z)\right).$$

Now performing the homothety

$$z = r_\varepsilon w\,,$$

the family $g_{\mathbf{b},\mathbf{hk}}$ has expansion

$$\begin{aligned}\omega_{g_{\mathbf{b},\mathbf{hk}}} \approx & i\partial\overline{\partial}\left(\frac{r_\varepsilon^2|w|^2}{2} + \psi_g\left(r_\varepsilon w\right)\right)\\ & + \left(\overline{b}_j^{2m}\varepsilon^{2m} + \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}}\right)c_{\Gamma_j} i\partial\overline{\partial}\left(-r_\varepsilon^{2-2m}|w|^{2-2m} + \frac{(m-1)\,s_g r_\varepsilon^{4-2m}}{2\,(m-2)\,m\,(m+1)}|w|^{4-2m}\right)\\ & + i\partial\overline{\partial}\left(\varepsilon^2\overline{b}_j^2\tilde{\chi}_{j,r_0}\psi_{\eta_j}\left(\frac{r_\varepsilon w}{\overline{b}_j\varepsilon}\right) + H^o_{h_j k_j}\left(w\right)\right) + \mathcal{O}\left(\varepsilon^{2m}r_\varepsilon^{4-2m}\right)\,.\end{aligned}$$

We set

$$\hat{b}_j := \sqrt[2m]{\overline{b}_j^{2m} + \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}}\varepsilon^{-2m}}\,.$$

Step 12: Now that we have the family $g_{\mathbf{b},\mathbf{hk}}$ we move to model spaces. We want to construct a family of cscK metrics that depend on ε and other suitable parameters on $X_{\frac{R_\varepsilon}{\hat{b}_j},j}$. This family of metrics must be a perturbation of the base metric by the $i\partial\overline{\partial}$ of a function $\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$:

$$\omega_{\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}} = \hat{b}_j^2\varepsilon^2\omega_{\eta_j} + \varepsilon^2 i\partial\overline{\partial}\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}\,.$$

Again, we want to construct this function in such a way that we can prescribe the behavior of the family of metrics at the truncation locus; to do this we will use inner biharmonic extensions of functions on the sphere that are invariant under the action of a finite subgroup of $SU(m)$. So our function $\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ will have the following form:

$$\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j} := \mathbb{J}_{\hat{b}_j} + \tilde{H}^I_{\tilde{h}_j\tilde{k}_j} + f^I_{\tilde{h}_j\tilde{k}_j}\,.$$

In the following four steps we explain how we construct the three components of $\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$.

Step 13: We start constructing the "skeleton" $\mathbb{J}_{\hat{b}_j}$ that is the most delicate step in building $\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$. Again, we do it in many steps. We would like to "bring to X_j" the non euclidean part ψ_g of the potential of g on $B_{r_0}(p_j)$ in such a way that, at the boundary, matches perfectly with the one on the corresponding piece of boundary of M_{r_ε}. To "bring it to X_j" we cutoff and rescale suitably ψ_g and we get

$$J_{\hat{b}_j}(x) := \frac{1}{\varepsilon^2}\tilde{\chi}_{R_0}\psi_g\left(\hat{b}_j\varepsilon x\right)$$

with $\tilde{\chi}_{R_0}$ smooth cutoff functions

$$\tilde{\chi}_{R_0}(p) : \begin{cases} 0 & p\in X_{\frac{R_0}{2\hat{b}_j},j}\\ 1 & p\in X_{\frac{3R_\varepsilon}{2\hat{b}_j},j}\setminus X_{\frac{R_0}{\hat{b}_j},j}\\ 0 & p\in X\setminus X_{\frac{2R_\varepsilon}{\hat{b}_j},j}\end{cases}$$

This naive skeleton isn't enough for our purposes, we need a more refined construction.

Step 14: We recall that ψ_g on $B_{r_0}(p_j)$ has the expansion

$$\psi_g(z) = \sum_{k=4}^{+\infty} p_k(z)$$

and the polynomials p_4, p_5 satisfy

$$\Delta^2 p_4 = -2s_g\,,$$
$$\Delta^2 p_5 = 0\,.$$

In light of this we'd like to find perturbations $u_{4,j}, u_{5,j}$ in some suitable spaces to get

$$\mathbb{L}_\eta\left(\chi_{R_0,j}p_4 - u_4\right) = -s_g\,,$$
$$\mathbb{L}_\eta\left(\chi_{R_0,j}p_5 - u_5\right) = 0\,,$$

with

$$\chi_{R_0,j}(p) : \begin{cases} 0 & p \in X_{\frac{R_0}{2\tilde{b}_j},j} \\ 1 & p \in X \setminus X_{\frac{R_0}{\tilde{b}_j},j} \end{cases}$$

The suitable spaces are weighted Hölder spaces of functions decaying at infinity with some particular order. It's a matter of computation to see that

$$\mathbb{L}_\eta\left(\chi_{R_0}p_4\right) = -s_g + |x|^{-2m}\left(\phi_{2,j} + \phi_{4,j}\right) + \mathcal{O}\left(|x|^{-2-2m}\right)\,,$$

with $\phi_{2,j}, \phi_{4,j}$ eigenfunctions of euclidean laplacian on the unit sphere relative to eigenvalues $-4m$ and $-8(m+1)$. We can make a first perturbation with $\chi_{R_0,j}|x|^{4-2m}\left(c_{2,j}\phi_{2,j} + c_{4,j}\phi_{4,j}\right)$ with $c_{2,j}, c_{4,j} \in \mathbb{R}$ well chosen such that

$$\mathbb{L}_\eta\left(\chi_{R_0}p_4 - \chi_{R_0,j}|x|^{4-2m}\left(c_{2,j}\phi_{2,j} + c_{4,j}\phi_{4,j}\right)\right) = -s_g + \mathcal{O}\left(|x|^{-2-2m}\right)\,.$$

We call $\tilde{u}_4$ the function $\tilde{u}_{4,j} = \chi_{R_0,j}|x|^{4-2m}\left(c_{2,j}\phi_{2,j} + c_4\phi_{4,j}\right)$. Now we would like to perturb $\chi_{R_0,j}p_4 - \tilde{u}_{4,j}$ with a function $\bar{u}_{4,j} \in C^{4,\alpha}_{2-2m+\delta'}$ with $\delta' \in (0,1)$ such that

$$\mathbb{L}_\eta\left(\chi_{R_0,j}p_4 - \tilde{u}_4 + \bar{u}_{4,j}\right) = -s_g\,.$$

Fredholm theory for weighted Hölder spaces tells us that we can't find such $\bar{u}_4$ unless the quantity

$$\int_X \left[\mathbb{L}_\eta\left(\chi_{R_0,j}p_4 - \tilde{u}_{4,j}\right) + s_g\right] d\mu_\eta$$

vanishes. It's again a matter of computations to see that the quantity above is not 0, indeed we can evaluate explicitly that integral and we get

$$\int_X \left[\mathbb{L}_\eta\left(\chi_{R_0,j}p_4 - \tilde{u}_{4,j}\right) + s_g\right] d\mu_\eta = -\frac{2c_\Gamma(m-1)^2\,\mu\left(S^{2m-1}\right)s_g}{m(m+1)|\Gamma|}\,.$$

We can overcome this obstruction adding a smooth function that decays at infinity like $|x|^{4-2m}$, more precisely we consider the function

$$\chi_{R_0,j}p_4 - \tilde{u}_{4,j} + \frac{c_{\Gamma_j}(m-1)s_g}{2(m-2)m(m+1)}\chi_{R_0,j}|x|^{4-2m}$$

We see immediately that

$$\mathbb{L}_{\eta}\left(\chi_{R_0,j}p_4-\tilde{u}_{4,j}+\frac{c_{\Gamma_j}(m-1)s_g}{2(m-2)m(m+1)}\chi_{R_0,j}|x|^{4-2m}\right)=-s_g+\mathcal{O}\left(|x|^{-2-2m}\right)$$

and

$$\int_X\left[\mathbb{L}_{\eta}\left(\chi_{R_0,j}p_4-\tilde{u}_4+\frac{c_{\Gamma}(m-1)s_g}{2(m-2)m(m+1)}\chi_{R_0,j}|x|^{4-2m}\right)+s_g\right]d\mu_\eta=0\,.$$

These are critical calculations and can be found in subsection 3.2.1. We can find $\bar{u}_{4,j}\in C^{4,\alpha}_{2-2m+\delta'}(X_j)$ such that

$$\mathbb{L}_{\eta}\left(\chi_{R_0,j}p_4-\tilde{u}_{4,j}+\frac{c_{\Gamma_j}(m-1)s_g}{2(m-2)m(m+1)}\chi_{R_0,j}|x|^{4-2m}+\bar{u}_{4,j}\right)=-s_g\,.$$

The price we pay for taking $\bar{u}_{4,j}$ with that particular decay is the appearance of an "undesired" function decaying like $|x|^{4-2m}$. We do the same procedure for the p_5 term and we see that we don't need to add "undesired" asymptotics to perturb it with a function $u_{5,j}$ that belongs to a "good space". Finally, setting

$$u_{4,j}:=\tilde{u}_{4,j}-\bar{u}_{4,j}\,,$$

we get our "skeleton" $\mathbb{J}_{\hat{b}_j}$

$$\begin{aligned}\mathbb{J}_{\hat{b}_j}:=&\hat{b}_j^4\varepsilon^2\left(\chi_{R_0,j}p_4(x)-u_{4,j}+\frac{c_{\Gamma}(m-1)s_g}{2(m-2)m(m+1)}\chi_{R_0,j}|x|^{4-2m}\right)\\&+\hat{b}_j^5\varepsilon^3\left(\chi_{R_0,j}p_5(x)-u_{5,j}\right)+\frac{1}{\varepsilon^2}\tilde{\chi}_{R_0}\left(\sum_{k=2}^{+\infty}p_{4+k}\left(\hat{b}_j\varepsilon x\right)\right)\,.\end{aligned}$$

Now we can finally see the relevance of the $|x|^{4-2m}$ asymptotics on M that we left unexplained in step 6. If we look at $\hat{b}^2\varepsilon^2\omega_{\eta_j}+\varepsilon^2 i\partial\overline{\partial}\mathbb{J}_{\hat{b}_j}$ and we make on $X_{\frac{R_\varepsilon}{\hat{b}_j},j}\setminus X_{\frac{R_\varepsilon}{2\hat{b}_j},j}$ the rescaling

$$x=\frac{z}{\hat{b}_j\varepsilon}\,,$$

we have

$$\begin{aligned}\hat{b}^2\varepsilon^2\omega_{\eta_j}+\varepsilon^2 i\partial\overline{\partial}\mathbb{J}_{\hat{b}_j}\approx& i\partial\overline{\partial}\left(\frac{|z|^2}{2}+\hat{b}_j^{2m}c_{\Gamma_j}\varepsilon^{2m}\left(-|z|^{2-2m}+\frac{(m-1)s_g}{2(m-2)m(m+1)}|z|^{4-2m}\right)\right)\\&+i\partial\overline{\partial}\left(\psi_g(z)+\hat{b}_j^2\varepsilon^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)\right)\,.\end{aligned}$$

Recalling the expansion of $g_{\mathbf{b},\mathbf{hk}}$ in step 12 we see that the first line of the right hand side above is identical to the first line of the right hand side. of the expansion in step 12. We made this "ad hoc" construction to perfectly match these specific quantities at the boundaries and obtain a "second order match". This step is subsection 3.2.1.

Step 15: We construct $\tilde{H}^I_{\tilde{h}_j\tilde{k}_j}$, that also in this case will prescribe the behavior at the boundary of the family $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$. More precisely, at $\partial X_{\frac{R_\varepsilon}{\hat{b}_j},j}$, again with the scaling

$$x=\frac{z}{\hat{b}_j\varepsilon}\,,$$

we want the following expansion

$$\begin{aligned}\omega_{\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}} \approx & i\partial\overline{\partial}\left(\frac{|z|^2}{2}+\hat{b}_j^{2m}c_{\Gamma_j}\varepsilon^{2m}\left(-|z|^{2-2m}+\frac{(m-1)\,s_g}{2\,(m-2)\,m\,(m+1)}|z|^{4-2m}\right)+\psi_g\,(z)\right)\\ &+\varepsilon^2 i\partial\overline{\partial}\left(\hat{b}_j^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)+\tilde{H}^I_{\tilde{h}_j\tilde{k}_j}\left(\frac{z}{r_\varepsilon}\right)\right)\,.\end{aligned}$$

We are asking, indeed, $f^I_{\tilde{h}_j\tilde{k}_j}$ to be much smaller than $\mathbb{J}_{\hat{b}_j}+\tilde{H}^I_{\tilde{h}_j\tilde{k}_j}$. Again, we will use euclidean biharmonic extensions to create $\tilde{H}^I_{\tilde{h}_j\tilde{k}_j}$, this time we will use inner biharmonic extensions. Euclidean inner biharmonic extension $H^I_{\tilde{h}_j\tilde{k}_j}$ of $\tilde{h}_j,\tilde{k}_j$ is the solution of the boundary value problem

$$\begin{cases}\Delta^2 H^I_{\tilde{h}_j\tilde{k}_j}=0 & w\in B_1\\ H^I_{\tilde{h}_j\tilde{k}_j}=\tilde{h}_j & w\in\partial B_1\\ \Delta H^I_{\tilde{h}_j\tilde{k}_j}=\tilde{k}_j & w\in\partial B_1\end{cases}$$

We'd like to define $\tilde{H}^I_{\tilde{h}_j\tilde{k}_j}$ "bringing to X_j" euclidean biharmonic extensions rescaling and cutting off them but it wouldn't be enough for our purposes, we need to make a more refined construction. We recall that Γ_j-invariant inner biharmonic extension have the form

$$H^I_{\tilde{h}_j,\tilde{k}_j}=\left(\tilde{h}_j^{(0)}-\frac{\tilde{k}_j^{(0)}}{4m}\right)+\frac{\tilde{k}_j^{(0)}}{4m}|w|^2+\sum_{\gamma=2}^{+\infty}\left(\left(\tilde{h}_j^{(\gamma)}-\frac{\tilde{k}_j^{(\gamma)}}{4(m+\gamma)}\right)|w|^\gamma+\frac{\tilde{k}_j^{(\gamma)}}{4(m+\gamma)}|w|^{\gamma+2}\right)\phi_\gamma\,.$$

We can apply to functions $|w|^2,|w|^2\phi_2,|w|^3\phi_3$ a procedure similar to that we used for p_4 and p_5 in step 14 to get rapidly decaying functions $u^0_{2,j},u^2_{2,j},u^3_{3,j}$ such that

$$\begin{aligned}\mathbb{L}_\eta\left(|x|^2-u^0_{2,j}\right)&=0\,,\\ \mathbb{L}_\eta\left(|x|^2\phi_2-u^2_{2,j}\right)&=0\,,\\ \mathbb{L}_\eta\left(|x|^3\phi_3-u^3_{3,j}\right)&=0\,.\end{aligned}$$

And we define

$$
\begin{aligned}
\tilde{H}^{I}_{\tilde{h}_j\tilde{k}_j} =& \left(\tilde{h}^{(0)}_j - \frac{\tilde{k}^{(0)}_j}{4m}\right) + \frac{\tilde{k}^{(0)}_j}{4mR_\varepsilon^2}\left(\chi_{R_0}|x|^2 - u^0_{2,j}\right)\\
&+\left[\left(\tilde{h}^{(2)}_j - \frac{\tilde{k}^{(2)}_j}{4(m+2)}\right)\frac{\chi_{R_0}|x|^2}{R_\varepsilon^2} + \frac{\tilde{k}^{(2)}_j}{4(m+2)R_\varepsilon^4}\chi_{R_0}|x|^4\right]\phi_2 - \left(\tilde{h}^{(2)}_j - \frac{\tilde{k}^{(2)}_j}{4(m+2)}\right)\frac{u^2_{2,j}}{R_\varepsilon^2}\\
&+\left[\left(\tilde{h}^{(3)}_j - \frac{\tilde{k}^{(3)}_j}{4(m+3)}\right)\frac{\chi_{R_0}|x|^3}{R_\varepsilon^3} + \frac{\tilde{k}^{(3)}_j}{4(m+3)R_\varepsilon^5}\chi_{R_0}|x|^5\right]\phi_3 - \left(\tilde{h}^{(3)}_j - \frac{\tilde{k}^{(3)}_j}{4(m+3)}\right)\frac{u^3_{3,j}}{R_\varepsilon^3}\\
&+\chi_{R_0}\left(\sum_{\gamma=4}^{+\infty}\left(\left(\tilde{h}^{(\gamma)}_j - \frac{\tilde{k}^{(\gamma)}_j}{4(m+\gamma)}\right)\left|\frac{x}{R_\varepsilon}\right|^\gamma + \frac{\tilde{k}^{(\gamma)}_j}{4(m+\gamma)}\left|\frac{x}{R_\varepsilon}\right|^{\gamma+2}\right)\phi_\gamma\right).
\end{aligned}
$$

We take $\left(\tilde{\mathbf{h}},\tilde{\mathbf{k}}\right)\in\mathcal{B}_\alpha$ such that $\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}}\right)\in\mathfrak{B}\left(\kappa,\beta,\sigma\right)$. This step is subsection 3.2.2.

Step 16: We construct the last block of $\mathbb{F}^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$: $f^{I}_{\tilde{h}_j\tilde{k}_j}$. We find it as a solution of a PDE in the weighted Hölder space $C^{4,\alpha}_{4-2m+\delta}\left(X_j\right)$ with $\delta\in(0,1)$. We now have our family of metrics $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ and if we make on $X_{\frac{R_\varepsilon}{\hat{b}_j},j}\setminus X_{\frac{R_\varepsilon}{2\hat{b}_j},j}$ the rescaling

$$
x=\frac{R_\varepsilon}{\hat{b}_j}w\,,
$$

we have

$$
\begin{aligned}
\omega_{\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}} =& i\partial\overline{\partial}\left(\frac{r_\varepsilon^2|w|^2}{2} + \hat{b}_j^{2m}c_{\Gamma_j}\varepsilon^{2m}\left(-r_\varepsilon^{2-2m}|w|^{2-2m} + \frac{(m-1)\,s_g r_\varepsilon^{4-2m}}{2\,(m-2)\,m\,(m+1)}|w|^{4-2m}\right)\right)\\
&+i\partial\overline{\partial}\psi_g\left(r_\varepsilon w\right)\\
&+\varepsilon^2 i\partial\overline{\partial}\left(\hat{b}_j^2\psi_{\eta_j}\left(\frac{R_\varepsilon w}{\hat{b}_j}\right) + H^{I}_{\tilde{h}_j\tilde{k}_j}\left(w\right)\right) + \mathcal{O}\left(r_\varepsilon^\sigma R_\varepsilon^{-2m+\delta}\right).
\end{aligned}
$$

This step is subsection 3.2.3.

Step 17: We now perform the procedure called “data matching”. This step is developed in sections 4.1 and 4.2. We have expansions

- of potentials of the family $g_{\mathbf{b},\mathbf{hk}}$ on the annuli $B_{2r_\varepsilon}\left(p_j\right)\setminus B_{r_\varepsilon}\left(p_j\right)$ and rescaling coordinates on $B_2\setminus B_1$;
- of potential of the families $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ on the annuli $X_{\frac{R_\varepsilon}{\hat{b}_j},j}\setminus X_{\frac{R_\varepsilon}{2\hat{b}_j},j}$ and rescaling coordinates on $B_1\setminus B_{\frac{1}{2}}$;

We can now translate the problem of gluing metrics along boundaries to that of gluing functions along spheres, indeed we have to find right $\mathbf{h},\mathbf{k},\tilde{\mathbf{h}},\tilde{\mathbf{k}}$ such that potentials of the various families glue as functions on $B_2\setminus B_{\frac{1}{2}}$ up to third derivatives. More precisely, let $\Psi^{o}_{\mathbf{b},\mathbf{hk}}$ be the potential of $g_{\mathbf{b},\mathbf{hk}}$ on the annuli $B_{2r_\varepsilon}(p_j) \setminus B_{r_\varepsilon}(p_j)$ and $\Psi^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ be the potential of $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ on the annuli $X_{\frac{R_\varepsilon}{\hat{b}_j},j} \setminus X_{\frac{R_\varepsilon}{2\hat{b}_j},j}$, if we can prove that at the respective boundaries the relations

$$\Psi^{o}_{\mathbf{b},\mathbf{hk}}(r_\varepsilon w) = \Psi^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right)$$

$$\partial_{|w|}\left[\Psi^{o}_{\mathbf{b},\mathbf{hk}}(r_\varepsilon w)\right] = \partial_{|w|}\left[\Psi^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right)\right]$$

$$\Delta\left[\Psi^{o}_{\mathbf{b},\mathbf{hk}}(r_\varepsilon w)\right] = \Delta\left[\Psi^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right)\right]$$

$$\partial_{|w|}\Delta\left[\Psi^{o}_{\mathbf{b},\mathbf{hk}}(r_\varepsilon w)\right] = \partial_{|w|}\Delta\left[\Psi^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right)\right]$$

are satisfied simultaneously, then automatically potentials glue analitically. We use the particular form of the above equations to define a continuous nonlinear operator

$$\mathcal{S} : \mathfrak{B}(\kappa, \beta, \sigma)^2 \to \mathcal{B}^2_\alpha$$

whose fixed points are the sets of boundary conditions that let us to glue our families of metrics. If we can show that $\mathcal{S}$ has a fixed point then we are done. To prove the existence of a fixed point we use a Picard iteration method and we can guarantee its convergence because of our careful construction of the families of metrics. Indeed the "second order match" at the boundary and the refined construction of the families of metrics translate to the fact that terms $\tilde{H}^{o}_{\mathbf{hk}}$ and $\tilde{H}^{I}_{\tilde{h}_j\tilde{k}_j}$, as functions of ε, dominate all the other terms (as functions of ε) that are not matched by construction. We prove that this is sufficient to guarantee the success of the iteration scheme and so to finish the proof of Theorem 1.7.

1.7 Notations and conventions

The proof of Theorem 1.7 is rather technical, we need to introduce a lot of notation and sometimes it can become very heavy. In this section we gather all the notations we use throughout the thesis. We assume Einstein summation convention.

1.7.1 General notation

- Γ, Γ_j will always be a finite subgroup of $SU(m)$.
- j will always be the letter relative to an index that goes from 1 to N
- the letter m will always denote the complex dimension of M and X_j and it is an integer greater or equal to 3.
- We denote with B_t the euclidean open ball of radius t, with $\bar{B}_t$ the closed euclidean ball of radius t and with B^*_t the open euclidean ball of radius t with the origin removed.
- We denote with A^Γ_t for $t > 0$ the set

$$A^\Gamma_t := \left(\bar{B}_t/\Gamma\right) \setminus \left(B_{\frac{t}{2}}/\Gamma\right) .$$

- w, $\left(w^1, \ldots, w^m\right)$ will always denote coordinates on $B_1 \setminus B_{1/2}$ or $B_2 \setminus B_1$.
- Δ will always denote the Euclidean Laplace operator
$$\Delta = 4\partial_i \overline{\partial}_i \,.$$
- $\phi_q, \tilde{\phi}_q$ will always be eigenfunctions of euclidean Laplace operator on the unit sphere S^{2m-1} relative to eigenvalue $-q\,(q+2m-2)$ that is
$$\Delta \phi_q = -q\,(q+2m-2)\,\phi_q$$
and we will always assume that they are invariant with respect to the action of a certain group Γ.
- $\left\{\bar{\phi}_{q,1}, \ldots, \bar{\phi}_{q,N_q}\right\}$ will always denote a $L^2\left(S^{2m-1}\right)$-orthonormal basis of the q-th eigenspace of $\Delta_{S^{2m-1}}$.
- ε will always be a reference small parameter.
- Always $h, h_j, \tilde{h}, \tilde{h}_j \in C^{4,\alpha}\left(\partial B_1\right)$ and $k, k_j, \tilde{k}, \tilde{k}_j \in C^{2,\alpha}\left(\partial B_1\right)$.
- If $f \in L^2\left(\partial B_1\right)$ then we denote with $f^{(q)}\phi_q$ the projection of f onto the q-th eigenspace of $\Delta_{S^{2m-1}}$ that is
$$f^{(q)}\phi_q := \sum_{l_q=1}^{N_q} (f)^{(q)}_{l_q}\, \bar{\phi}_{q,l_q} \,.$$
- If $f \in C^{0,\alpha}\left(\partial B_1\right)$ then we denote with $f^{(0)}$ its mean and with $f^{(\dagger)}$
$$f^{(\dagger)} := f - f^{(0)} \,.$$
If $f \in C^{0,\alpha}\left(B_1 \setminus B_{1/2}\right)$ or $f \in C^{0,\alpha}\left(B_2 \setminus B_1\right)$ we denote with $f^{(0)}$
$$f^{(0)}\left(|w|\right) := \int_{S^{2m-1}} f\left(|w|, \frac{w}{|w|}\right) d\mu_0 \,,$$
and $f^{(\dagger)}$
$$f^{(\dagger)} := f - f^{(0)} \,.$$
- $H^o_{hk}\left(w\right)$ will always denote outer euclidean biharmonic extension of functions on the sphere $h \in C^{4,\alpha}\left(\partial B_1\right)$,$k \in C^{2,\alpha}\left(\partial B_1\right)$.
- $H^I_{hk}\left(w\right)$ will always denote inner euclidean biharmonic extension of functions on the sphere $h \in C^{4,\alpha}\left(\partial B_1\right)$, $k \in C^{2,\alpha}\left(\partial B_1\right)$.
- with χ_1 we denote a smooth cutoff function on $[0, +\infty)$
$$\chi_1\left(t\right) = \begin{cases} 0 & t \in \left[0, \frac{1}{2}\right] \\ 1 & t \in [1, +\infty) \end{cases}$$
- with χ_2 we denote a smooth cutoff function on $[0, +\infty)$
$$\chi_2\left(t\right) = \begin{cases} 1 & [0, 1] \\ 0 & t \in [2, +\infty) \end{cases}$$

- For every tensor T of type
$$T = T_{i\bar{j}} dz_i \otimes d\bar{z}_j$$
on a complex manifold M and every $u \in \mathcal{C}^\infty(M)$ we indicate
$$\operatorname{tr}\left(\partial\overline{\partial}u \cdot T\right) = g^{i\bar{l}} g^{k\bar{j}} \partial_i \bar{\partial}_j u \, T_{k\bar{l}}.$$

1.7.2 Spaces and sets

On M

- (M, g) will always be a compact cscK orbifold with isolated singularities.
- $\mathbf{p}$ is the (finite) set of $SU(m)$ singular points of M and $\sharp\mathbf{p} = N$.
- M_{r_ε} will denote the set
$$M_{r_\varepsilon} := M \setminus \bigcup_{p \in \mathbf{p}} B_{r_\varepsilon}(p) .$$
- $C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})$, $C^{0,\alpha}_{\tau-2m}(M_{\mathbf{p}})$ with $\tau \in (0,1)$ will always denote weighted Hölder spaces on M with points $\mathbf{p}$ removed.
- $\mathcal{D}$ will denote the (finite dimensional) deficiency space for $\mathbb{L}_g$ that we will use to extend $C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})$ and its generic element is denoted as H_f.
- We will indicate with $\mathcal{B}_\alpha$
$$\mathcal{B}_\alpha := \left\{ (\mathbf{h}, \mathbf{k}) \in C^{4,\alpha}(\partial B_1)^N \times C^{2,\alpha}(\partial B_1)^N \,|\, h_j, k_j \text{ are } \Gamma_j - \text{invariant} \right\}$$
except for section 3.2 in which (with abuse of notation) we indicate $\mathcal{B}_\alpha$
$$\mathcal{B}_\alpha := \left\{ \left(\tilde{h}, \tilde{k}\right) \in C^{4,\alpha}(\partial B_1) \times C^{2,\alpha}(\partial B_1) \,|\, \tilde{h}, \tilde{k} \text{ are } \Gamma - \text{invariant} \right\} .$$
- We indicate with $\mathfrak{B}(\kappa, \beta, \sigma) \subset \mathcal{B}_\alpha$ the set of $(\mathbf{h}, \mathbf{k}) \in \mathcal{B}_\alpha$ such that
$$\left\| \left(\mathbf{h}^{(0)}, \mathbf{k}^{(0)}\right) \right\|_{\mathcal{B}_\alpha} \leq \kappa r_\varepsilon^\beta ,$$
$$\left\| \left(\mathbf{h}^{(\dagger)}, \mathbf{k}^{(\dagger)}\right) \right\|_{\mathcal{B}_\alpha} \leq \kappa r_\varepsilon^\sigma .$$

On X

- (X_j, η_j),(X, η), if not stated otherwise, will always be an ALE Kähler Ricci-Flat space with projection
$$\pi_j : X_j \to \mathbb{C}^m / \Gamma_j .$$
- $X_{\frac{R_\varepsilon}{b_j}, j}$, $X_{\frac{R_\varepsilon}{b}}$ denote the manifold with boundary
$$X_{\frac{R_\varepsilon}{b_j}, j} := X_j \setminus \pi_j^{-1}\left(B_{\frac{R_\varepsilon}{b_j}}\right) .$$

- $C^{4,\alpha}_{4+\delta-2m}(X_j)$, $C^{0,\alpha}_{\delta-2m}(X_j)$ with $\delta \in (0,1)$ will always denote weighted Hölder spaces on X_j.

- We indicate with $\mathfrak{B}(\kappa, \beta', \sigma') \subset \mathcal{B}_\alpha$ the set of $\left(\tilde{h}_j^{(0)}, \tilde{k}_j^{(0)}\right) \in \mathcal{B}_\alpha$ such that

$$\left\| \left(\tilde{h}_j^{(0)}, \tilde{k}_j^{(0)}\right) \right\|_{\mathcal{B}_\alpha} \leq \kappa R_\varepsilon^{-\beta'} ,$$

$$\left\| \left(\tilde{h}_j^{(\dagger)}, \tilde{k}_j^{(\dagger)}\right) \right\|_{\mathcal{B}_\alpha} \leq \kappa R_\varepsilon^{-\sigma'} .$$

1.7.3 Parameters

On M

- r_0 will always be a small number independent of ε.

- $r_\varepsilon = \varepsilon^{\frac{2m-1}{2m+1}}$.

- $\tau \in \left(0, \frac{1}{(m+2)^2}\right)$.

- $r_\varepsilon^\sigma = \varepsilon^{2m+4} r_\varepsilon^{-2-2m-\tau}$.

- $r_\varepsilon^\beta = \varepsilon^{4m+2} r_\varepsilon^{-4m-\tau}$.

- b_j, $\mathbf{b} := (b_1, \ldots, b_N)$ will always be positive numbers independent of ε.

- $\tilde{\mathbf{b}}$, $\tilde{b}_j$ will be positive numbers small perturbations of the above $\mathbf{b}$ depending on ε.

- $\bar{\mathbf{b}}$, $\bar{b}_j$ will be positive numbers depending on $\tilde{\mathbf{b}}$

$$\bar{b}_j := \sqrt[2m]{\tilde{b}_j} .$$

- $\hat{b}_j = \sqrt[2m]{\bar{b}_j^{2m} + \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \varepsilon^{-2m}}$ with $\hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}}$ coming from the term H_f of $f^o_{\mathbf{b},\mathbf{hk}}$.

On X

- R_0 will always be a big quantity independent of ε.

- $R_\varepsilon = \frac{r_\varepsilon}{\varepsilon}$.

- $R_\varepsilon^{-\sigma'} = \frac{r_\varepsilon^\sigma}{\varepsilon^2}$.

- $R_\varepsilon^{-\beta'} = \frac{r_\varepsilon^\beta}{\varepsilon^2}$.

- $\delta \in \left(0, \frac{1}{(m+2)^2}\right)$.

1.7.4 Coordinates, functions and tensors

On M

- z, $\left(z^1, \dots, z^m\right)$ will always denote local coordinates on M.
- s_g will always denote the scalar curvature of M with respect to g.
- In normal coordinates around a point $p \in M$ the metric g has expansion

$$\omega_g = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + \psi_g(z)\right),$$

moreover

$$\psi_g(z) = \sum_{k=4}^{+\infty} p_k(z),$$

with p_k homogeneous polynomials of degree k and

$$p_4(z) = -\frac{1}{4} R_{i\bar{j}k\bar{l}}(p)\, z^i \overline{z^j} z^k \overline{z^l}$$

with $R_{i\bar{j}k\bar{l}}(p)$ the Riemann curvature tensor.

- φ, φ_k will always denote functions in $\ker(\mathbb{L}_g)$.
- With χ_{r_0} we denote a smooth cutoff function

$$\chi_{r_0}(q) := \begin{cases} 1 & q \in B_{r_0}(p) \\ 0 & q \in M \setminus B_{2r_0}(p) \end{cases}$$

for any $p \in M$.

- With χ_{j,r_0} we denote a smooth cutoff function

$$\chi_{j,r_0}(q) := \begin{cases} 1 & q \in B_{r_0}(p_j) \\ 0 & q \in M \setminus B_{2r_0}(p_j) \end{cases}$$

for any $p_j \in \mathbf{p}$.

- With $\tilde{\chi}_{j,r_0}$ we denote a smooth cutoff function

$$\tilde{\chi}_{j,r_0}(q) := \begin{cases} 0 & q \in B_{\frac{r_\varepsilon}{3}}(p_j) \\ 1 & q \in B_{r_0}(p_j) \setminus B_{\frac{r_\varepsilon}{2}}(p_j) \\ 0 & q \in M \setminus B_{2r_0}(p_j) \end{cases}$$

for any $p_j \in \mathbf{p}$.

- $g_{\mathbf{b},\mathbf{hk}}$ will denote the family of cscK metric on M_{r_ε} depending on $\mathbf{b}$, $\mathbf{h}, \mathbf{k}$, moreover

$$\omega_{g_{\mathbf{b},\mathbf{hk}}} = \omega_g + i\partial\overline{\partial}\mathbb{F}^o_{\mathbf{b},\mathbf{hk}}.$$

- the function $\mathbb{F}^o_{\mathbf{b},\mathbf{hk}}$ decomposes as

$$\mathbb{F}^o_{\mathbf{b},\mathbf{hk}} = \mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}.$$

- the function $f^{o}_{\mathbf{b},\mathbf{hk}}$ decomposes as

$$f^{o}_{\mathbf{b},\mathbf{hk}} = \tilde{f}^{o}_{\mathbf{b},\mathbf{hk}} + H_f$$

with $\tilde{f}^{o}_{\mathbf{b},\mathbf{hk}} \in C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})$, $H_f \in \mathcal{D}$ and for any $p_j \in \mathbf{p}$, on $B_{r_\varepsilon}(p)$, we have the expansion

$$H_f = c_{\Gamma_j} \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \left(-|z|^{2-2m} + \frac{(m-1)\, s_g}{2\,(m-2)\, m\,(m+1)} |z|^{4-2m} \right) + \mathcal{O}\left(|z|^{4-2m}\right) .$$

- $\Psi^{o}_{\mathbf{b},\mathbf{hk}}$ is the potential of the family of metrics $g_{\mathbf{b},\mathbf{hk}}$ on $B_{2r_\varepsilon}(p) \setminus B_{r_\varepsilon}(p)$ for $p \in \mathbf{p}$.

On X

- x, $\left(x^1, \ldots, x^m\right)$ will always denote local coordinates on X_j, X.
- In coordinates "at infinity" on X the metric η has expansion

$$\omega_\eta = i\partial\overline{\partial}\left(\frac{|x|^2}{2} - c_\Gamma |x|^{2-2m} + \psi_\eta(x) \right) ,$$

moreover

$$\psi_\eta(x) = \mathcal{O}\left(|x|^{-2m}\right) .$$

- With $\chi_{R_0,j}$ we denote a smooth cutoff function

$$\chi_{R_0,j}(q) := \begin{cases} 0 & q \in X_{\frac{R_0}{2\hat{b}_j}} \\ 1 & q \in X \setminus X_{\frac{R_0}{\hat{b}_j}} \end{cases}$$

- With $\tilde{\chi}_{R_0,j}$ we denote a smooth cutoff function

$$\tilde{\chi}_{R_0,j}(q) := \begin{cases} 0 & q \in X_{\frac{R_0}{2\hat{b}_j},j} \\ 1 & q \in X_{\frac{3R_\varepsilon}{2\hat{b}_j}} \setminus X_{\frac{R_0}{\hat{b}_j},j} \\ 0 & q \in X \setminus X_{\frac{2R_\varepsilon}{\hat{b}_j},j} \end{cases}$$

- $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ will denote the family of cscK metric on $X_{\frac{R_\varepsilon}{\hat{b}_j},j}$ depending on $\hat{b}_j$, $\tilde{h}_j, \tilde{k}_j$, moreover

$$\omega_{\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}} = \hat{b}_j^2 \varepsilon^2 \omega_{\eta_j} + i\partial\overline{\partial}\mathbb{F}^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$$

- the function $\mathbb{F}^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ decomposes as

$$\mathbb{F}^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j} = \mathbb{J}_{\hat{b}_j} + \tilde{H}^{I}_{\tilde{h}_j\tilde{k}_j} + f^{I}_{\tilde{h}_j\tilde{k}_j}$$

- $\Psi^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ is the potential of the family of metrics $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ on $X_{\frac{R_\varepsilon}{\hat{b}_j},j} \setminus X_{\frac{R_\varepsilon}{2\hat{b}_j},j}$

1.7.5 Operators

On M

- Δ_g will always denote the Laplace-Beltrami operator induced by g on M and we use the definition
$$\Delta_g = 2g^{i\bar{j}}\partial_i\overline{\partial}_j \,.$$
- $\mathbb{L}_g$ will always denote Lichnerowicz operator induced by g on M, it is defined
$$\mathbb{L}_g = \frac{1}{2}\Delta_g^2 + 2\left\langle \mathrm{Ric}_g, i\partial\overline{\partial}\cdot\right\rangle_g \,.$$

On X

- $\Delta_\eta, \Delta_{\eta_j}$ will always denote the Laplace-Beltrami operator induced by η, η_j on X, X_j and we use the definition
$$\Delta_\eta = 2\eta^{i\bar{j}}\partial_i\overline{\partial}_j \,.$$
- $\mathbb{L}_\eta$ will always denote Lichnerowicz operator induced by η on X, it is defined
$$\mathbb{L}_\eta = \frac{1}{2}\Delta_\eta^2 + 2\left\langle \mathrm{Ric}_\eta, i\partial\overline{\partial}\cdot\right\rangle_\eta$$
and on X reduces to
$$\mathbb{L}_\eta = \frac{1}{2}\Delta_\eta^2 \,.$$

Chapter 2

Linear analysis

Since we are following the strategy of proof of [AP06],[AP09] the first thing we have to understand is under which conditions the operators $\mathbb{L}_g$ on a compact cscK orbifold with isolated singularities and $\mathbb{L}_\eta$ on an ALE space are invertible. In this chapter we will study the invertibility properties for these linear operators.

2.1 Analysis on cscK orbifolds

2.1.1 The equation $\mathbb{L}_g u = f$

One of the main issues in the gluing construction, in presence of holomorphic vector fields, is that $\dim \ker(\mathbb{L}_g) > 1$, that is there are nonconstant functions in $\ker(\mathbb{L}_g)$. The operator $\mathbb{L}_g$ is self adjoint and so if $f \in L^2(M)$ then the equation

$$\mathbb{L}_g u = f$$

has a solution iff f is L^2-orthogonal to $\ker(\mathbb{L}_g)$. We want to study the case in which f is not in L^2 but blows up at some particular rate. We start from a local model of solution and then we try to globalize it.

Definition 2.1. Let (M, g) be a Kähler orbifold and let $p_1, \ldots, p_N \in M$ be its isolated singular points with neighborhoods $U(p_j)$ biholomorphic to B_r/Γ_j with Γ_j finite subgroups of $U(m)$ that act linearly and freely on S^{2m-1}. We say that $f \in C^0(M)$ belongs to $C^{k,\alpha}(M)$ if

$$f \in C^{k,\alpha}(M \setminus \{p_1, \ldots, p_N\})$$

and

$$f \circ \pi_{\Gamma_j} \in C^{k,\alpha}(B_r)$$

with π_{Γ_j} the quotient map

$$\pi_{\Gamma_j} : B_r \longrightarrow B_r/\Gamma_j\,.$$

A consequence of the above definition is the following lemma.

Lemma 2.1. *Let (M, g) be a Kähler orbifold with isolated singularities and $f \in C^3(M)$. Then, at a singular point p, $f \circ \pi_{\Gamma_p}$ has a Taylor expansion of type*

$$f \circ \pi_{\Gamma_p}(z) = f(p) + \partial_i \overline{\partial}_j f \circ \pi_{\Gamma_p}(0)\, z^i \overline{z^j} + \frac{1}{2}\left[\partial_i \partial_j f \circ \pi_{\Gamma_p}(0)\, z^i z^j + \overline{\partial_i \partial_j f \circ \pi_{\Gamma_p}(0)\, z^i z^j}\right] + \mathcal{O}\left(|z|^3\right)\,.$$

Proof. Since $f \circ \pi_{\Gamma_p} \in C^3(B_r)$, then has a Taylor expansion

$$\begin{aligned} f \circ \pi_{\Gamma_p}(z) =& f \circ \pi_{\Gamma_p}(0) + \left[\partial_i f \circ \pi_{\Gamma_p}(0) z^i + \overline{\partial_i f \circ \pi_{\Gamma_p}(0) z^i}\right] \\ &+ \partial_i \overline{\partial}_j f \circ \pi_{\Gamma_p}(0)\, z^i \overline{z^j} \\ &+ \frac{1}{2}\left[\partial_i \partial_j f \circ \pi_{\Gamma_p}(0)\, z^i z^j + \overline{\partial_i \partial_j f \circ \pi_{\Gamma_p}(0)\, z^i z^j}\right] + \mathcal{O}\left(|z|^3\right) . \end{aligned}$$

Since $f \circ \pi_{\Gamma_p}$ is Γ_p-invariant, then the linear term of the expansion

$$\partial_i f \circ \pi_{\Gamma_p}(0) z^i + \overline{\partial_i f \circ \pi_{\Gamma_p}(0) z^i}$$

must be Γ_p-invariant too, but there aren't Γ_p-invariant linear functions and so the lemma follows immediately. □

General L^p Theory for $\mathbb{L}_g$

The following results are well known since they are a slighter extension of those present in [LS94] but for the sake of clearness and completeness we give the proofs. From now on (M, g) will be a compact cscK orbifold with isolated singularities.

Theorem 2.1. *Let (M, g) be a compact cscK orbifold with isolated singularities, then the operator*

$$\mathbb{L}_g : W^{4,2}(M) \to L^2(M)$$

is Fredholm with index 0 and

$$\ker(\mathbb{L}_g) = coker(\mathbb{L}_g) .$$

So there exists a continuous map

$$S_{\mathbb{L}_g} : L^2(M) / \ker(\mathbb{L}_g) \to W^{4,2}(M) / \ker(\mathbb{L}_g)$$

such that

$$\mathbb{L}_g \circ S_{\mathbb{L}_g} = I_{L^2(M)/\ker(\mathbb{L}_g)} \qquad S_{\mathbb{L}_g} \circ \mathbb{L}_g = I_{W^{4,2}(M)/\ker(\mathbb{L}_g)} .$$

Proof. We know by [LS94] that $\mathbb{L}_g$ has closed graph and

$$\ker(\mathbb{L}_g) = \ker\left(\overline{\partial}\partial^\sharp\right) = \operatorname{span}\{1, \varphi_1, \ldots, \varphi_N\}$$

with φ_i smooth functions. Let $\varphi \in \ker(\mathbb{L}_g)$ then $\forall u \in W^{4,2}(M)$

$$\int_M \mathbb{L}_g u \varphi d\mu_g = \int_M \left\langle \overline{\partial}\partial^\sharp u, \overline{\partial}\partial^\sharp \varphi \right\rangle_g = 0$$

so $\ker(\mathbb{L}_g) \subset \operatorname{coker}(\mathbb{L}_g)$. Now let $\psi \in \operatorname{coker}(\mathbb{L}_g) \cap \ker(\mathbb{L}_g)^\perp$ with $\|\psi\|_{L^2(M)} = 1$, there exist a sequence in $\{\psi_k\}_{k\in\mathbb{N}} \in C^\infty(M) \cap \ker(\mathbb{L}_g)^\perp$ with $\|\psi_k\|_{L^2(M)} = 1$ such that

$$\|\psi_k - \psi\|_{L^2(M)} < \frac{1}{k} \quad k \to +\infty .$$

But then $\forall u \in W^{4,2}(M)$

$$\begin{aligned} \int_M \mathbb{L}_g u\, \psi_k\, d\mu_g &= \int_M \mathbb{L}_g u \psi d\mu_g + \int_M u \mathbb{L}_g(\psi_k - \psi)\, d\mu_g \\ &= \int_M \mathbb{L}_g u\, (\psi_k - \psi)\, d\mu_g \end{aligned}$$

and so

$$\left|\int_M \mathbb{L}_g u\psi_k d\mu_g\right| \leq \int_M |\mathbb{L}_g u| \, |\psi_k - \psi| \, d\mu_g \leq \|\mathbb{L}_g u\|_{L^2(M)} \|\psi_k - \psi\|_{L^2(M)} \leq \frac{\|\mathbb{L}_g u\|_{L^2(M)}}{k} .$$

So the functional

$$T_{\psi_k} : \mathrm{im}\,(\mathbb{L}_g) \to \mathbb{R} ,$$

$$T_{\psi_k}(h) := \int_M h\psi_k ,$$

is linear continuous with $\|T_{\psi_k}\| \leq \frac{1}{k}$. By Hahn-Banach theorem we can extend it to a linear continuous functional on $L^2(M)$ with the same norm. By uniform boundedness principle we have a limit functional T for the sequence T_{ψ_k} s.t.

$$\|T\| \leq \liminf_{k\to+\infty} \|T_{\psi_k}\| = 0$$

By Riez representation theorem we have that

$$T(h) = \int_M h\bar{\psi} d\mu_g$$

and so $\bar{\psi} = 0$. But

$$1 = \int_M \psi^2 = \lim_{k\to+\infty} \int_M \psi_k\psi = \lim_{k\to+\infty} T_{\psi_k}(\psi) = \int_M \psi\bar{\psi} = 0$$

contraddiction. We conclude that

$$\mathrm{coker}\,(\mathbb{L}_g) = \ker\,(\mathbb{L}_g) .$$

□

Proposition 2.1. *Let $f \in L^p(M)/\ker(\mathbb{L}_g)$ with $1 < p < +\infty$, then there exist*

$$u \in W^{4,p}(M)/\ker(\mathbb{L}_g)$$

such that

$$\|u\|_{W^{4,p}(M)} \leq C \|f\|_{L^p(M)} .$$

Proof. Let $\{f_n\}_{n\in\mathbb{N}} \in L^2(M)/\ker(\mathbb{L}_g)$ such that

$$f_n \overset{L^p(M)}{\longrightarrow} f .$$

Then by Theorems A.14 and A.15 we have $\{u_n\}_{n\in\mathbb{N}} \in W^{4,p}(M)$

$$\|u_n\|_{W^{4,p}(M)} \leq C \|f_n\|_{L^p(M)} \leq 2C \|f\|_{L^p(M)}$$

so by compactness of the embedding

$$W^{4,p}(M) \hookrightarrow L^p(M)$$

we have a limit $u \in L^p(M)/\ker(\mathbb{L}_g)$

$$u_n \longrightarrow u$$

that satisfy

$$\begin{aligned}\int_M u\mathbb{L}_g v d\mu_g &= \lim_{n\to+\infty}\int_M u_n \mathbb{L}_g v d\mu_g \\ &= \lim_{n\to+\infty}\int_M \mathbb{L}_g u_n v d\mu_g \\ &= \lim_{n\to+\infty}\int_M f_n v d\mu_g \\ &= \int_M f v d\mu_g\,.\end{aligned}$$

So it is a distributional solution of the equation

$$\mathbb{L}_g u = f\,.$$

Then by Theorem A.14

$$\|u\|_{W^{4,p}(M)} \leq C\left(\|f\|_{L^p(M)} + \|u\|_{L^p(M)}\right)$$

and by Theorem A.15

$$\|u\|_{W^{4,p}(M)} \leq C'\,\|f\|_{L^p(M)}\,.$$

□

We introduce now weighted Hölder spaces on manifolds.

Definition 2.2. Let (M,g) be a compact Riemannian manifold, let $p_1,\dots,p_n \in M$ and

$$M_{\mathbf{p}} = M \setminus \{p_1\dots,p_N\}\,.$$

We take balls $B_{r_0}(p_k)$, Riemannian normal coordinates on them centered at p_k's and we define

$$M_{r_0} := M \setminus \left(\bigcup_{k=1}^N B_{r_0(p_k)}\right)\,.$$

Let $\delta \in \mathbb{R}$, $\alpha \in (0,1)$, we define the weighted Hölder space $C^{k,\alpha}_\delta(M_{\mathbf{p}}) \subset C^{k,\alpha}_{loc}(M_{\mathbf{p}})$ of functions $f \in C^{k,\alpha}_{loc}(M_{\mathbf{p}})$ such that

$$\|f\|_{C^{k,\alpha}_\delta(M_{\mathbf{p}})} := \|f\|_{C^{k,\alpha}_\delta\left(M_{r_0}\right)} + \sup_{\substack{1\leq j\leq n\\ 0<r\leq r_0}} r^{-\delta}\left\|\,f(r\cdot)|_{B_{r_0}(p_j)}\right\|_{C^{k,\alpha}(B_2\setminus B_1)} < +\infty\,.$$

Remark 2.1. Weighted Hölder spaces are Banach spaces with respect to the norm defined above.

We recall a fundamental fact whose proof follows immediately using [Pac08, Proposition 4.0.1] and standard Schauder estimates.

Theorem 2.2. *Let $n \geq 3$, $B_1 \subseteq \mathbb{R}^n$ the unit ball and $\delta \in (2-n,0)$. Let $f \in C^{0,\alpha}_{\delta-2}\left(\bar{B}_1^*\right)$ then there exist a unique $u \in C^0_\delta\left(\bar{B}_1^*\right) \cap C^{2,\alpha}_\delta\left(\bar{B}_{1/2}^*\right)$ that solves the problem*

$$(*)\begin{cases}\Delta u = f\\ u|_{\partial B_1} = 0\end{cases}$$

and satisfy the estimates

$$\|u\|_{C_\delta\left(\bar{B}_1^*\right)} \leq C\left(n,\delta\right)\|f\|_{C_{\delta-2}\left(\bar{B}_1^*\right)} \,,$$
$$\|u\|_{C_\delta^{2,\alpha}\left(\bar{B}_{\frac{1}{2}}^*\right)} \leq C\left(n,\delta\right)\|f\|_{C_{\delta-2}^{0,\alpha}\left(\bar{B}_1^*\right)} \,.$$

We now prove the kind of result of Proposition 2.1 for weighted spaces $C_\delta^{4,\alpha}(M)$, indeed we prove the following proposition.

Proposition 2.2. *Let* $\delta \in (4-2m, 0)$ *and let* $f \in C_{\delta-4}^{0,\alpha}(M)$*, if* f *is* L^2*-orthogonal to* $\ker(\mathbb{L}_g)$ *then there exist a unique* $u \in C_\delta^{4,\alpha}(M)$ *that is* L^2*-orthogonal to* $\ker(\mathbb{L}_g)$ *such that*

$$\mathbb{L}_g u = f$$

and satisfies

$$\|u\|_{C_\delta^{4,\alpha}(M_p)} \leq C\left(\delta, m, g, M\right)\|f\|_{C_{\delta-4}^{0,\alpha}(M)} \,.$$

Proof. We note that $C_{\delta-4}^{0,\alpha}(M)$ embeds in $L^p(M)$ for some $1 < p < \infty$, then the existence of a solution u is assured by Proposition 2.1, we only have to prove the estimates. We pick a coordinate (Normal Kähler) ball $B_r(p)$ around each point on which f blows up. To reach our goal we construct, by hands, local solutions in these neighborhoods in such a way we have refined (local) estimates. We then show that the solution u and local solutions we constructed differ by $C^{0,\alpha}$ functions and so we can conclude using standard Schauder estimates. Using Theorem 2.2 we first solve

$$(*_0): \begin{cases} \Delta^2 v_0 = 2f \\ v_0 \mid_{\partial B_r} = 0 \\ \Delta v_0 \mid_{\partial B_r} = 0 \end{cases}$$

and we get $v_0 \in C_\delta\left(B_r(p)\setminus\{p\}\right) \cap C_{loc}^{4,\alpha}\left(B_r(p)\setminus\{p\}\right)$. Now we set up a family of auxiliary problems for $k \in \mathbb{N}^+$

$$(*_k): \begin{cases} \Delta^2 v_k = \left[2\mathbb{L}_g - \Delta^2\right](v_{k-1}) \\ v_k \mid_{\partial B_r} = 0 \\ \Delta v_k \mid_{\partial B_r} = 0 \end{cases}$$

and we get $v_k \in C_{\delta+2k}\left(B_r(p)\setminus\{p\}\right) \cap C_{loc}^{4,\alpha}\left(B_r(p)\setminus\{p\}\right)$. If we set

$$u_N = \sum_{k=0}^{N} (-1)^k v_k \,,$$

then we have

$$\mathbb{L}_g u_N = (-1)^N \left[\mathbb{L}_g - \frac{\Delta^2}{2}\right](v_{N-1})$$

and for $N > 1 - \frac{\delta}{2}$ we have that $\mathbb{L}_g u_N \in C^{0,\alpha}\left(B_r(p)\right)$. Moreover by Theorem 2.2 we have the estimate

$$\|u_N\|_{C_\delta^{4,\alpha}\left(B_{2^{-N-1}r}(p)\setminus\{p\}\right)} \leq C\left(g,r,m,\delta\right)\|f\|_{C_{\delta-4}^{0,\alpha}\left(B_r(p)\setminus\{p\}\right)} \,.$$

Let χ be a cutoff function supported on $B_{2^{-N-2}r}(p)$, we have

$$\|\chi u_N\|_{C_\delta^{4,\alpha}(M)} \leq C\left(\delta, m, g, M, r\right)\|\chi f\|_{C_{\delta-4}^{0,\alpha}(M)} \,,$$

moreover we define

$$\tilde{u}_N = \chi u_N - \sum_{i=1}^{d} \langle \chi u_N, \varphi_i \rangle_{L^2(M)} \varphi_i$$

and we have

$$\|\tilde{u}_N\|_{C^{4,\alpha}_{\delta}(M)} \leq C\left(\delta, m, g, M, r\right) \|f\|_{C^{0,\alpha}_{\delta-4}(M)} \,.$$

We have by construction that

$$\mathbb{L}_g\left(u - \tilde{u}_N\right) = f - \mathbb{L}_g\left(\tilde{u}_N\right) \qquad \text{and} \qquad f - \mathbb{L}_g\left(\tilde{u}_N\right) \in C^{0,\alpha}\left(M\right) \,,$$

moreover $u - \tilde{u}_N$ is L^2-orthogonal to $\ker\left(\mathbb{L}_g\right)$ and so we have

$$\|u - \tilde{u}_N\|_{C^{4,\alpha}(M)} \leq C\left(\delta, m, g, M, r\right) \|f - \mathbb{L}_g\left(\tilde{u}_N\right)\|_{C^{0,\alpha}(M)} \leq C\left(\delta, m, g, M, r\right) \|f\|_{C^{0,\alpha}_{\delta-4}(M)} \,.$$

Summing up what we have done until now we have

$$\|u\|_{C^{4,\alpha}_{\delta}(M)} \leq \|\tilde{u}_N\|_{C^{4,\alpha}_{\delta}(M)} + \|u - \tilde{u}_N\|_{C^{4,\alpha}_{\delta}(M)} \leq C\left(\delta, m, g, M, r\right) \|f\|_{C^{0,\alpha}_{\delta-4}(M)} \,.$$

□

The operator $\mathbb{L}_g$ on $\mathcal{D}'\left(M\right)$

In the sequel we will need to solve the equations of type

$$\mathbb{L}_g\left(u\right) = \mu \,,$$

with $\mu \in \mathcal{D}'\left(M\right)$ with its support $\operatorname{supp}\left(\mu\right)$ consisting of isolated points. Our μ will be tipically Dirac delta function and laplacians of Dirac delta functions. The solution u, provided it exists, turns out to be a smooth function on $M \setminus \operatorname{supp}\left(\mu\right)$ and on $\operatorname{supp}\left(\mu\right)$ blows up at some rate. To get informations on these blow up rates we now want to construct "by hand" local approximate solutions for the equation $\mathbb{L}_g\left(u\right) = 0$ that will enable us to get refined estimates on the blow up behavior of u.

Definition 2.3. We say that a function $u \in C^{4,\alpha}_{loc}\left(B_r\left(p\right) \setminus \{p\}\right)$ is an approximate solution of

$$\mathbb{L}_g\left(u\right) = 0 \qquad \text{on } B_r\left(p\right) \setminus \{p\}$$

if, on the whole $B_r\left(p\right)$, we have

$$\mathbb{L}_g\left(u\right) = \mu_p + f$$

with $\mu_p \in \mathcal{D}'\left(B_r\left(p\right)\right)$, $\operatorname{supp}\left(\mu_p\right) = \{p\}$ and $f \in C^{0,\alpha}\left(B_r\left(p\right)\right)$.

By the shape in local coordinates of the operator $\mathbb{L}_g$ and the typical distributions we want to consider we are led to analyze two particular rates of blow up, that is $4 - 2m$ and $2 - 2m$. The following proposition is implicit in the work [AP09], we give here a proof for the sake of completeness.

Proposition 2.3. *Let $p \in M$, $m \geq 3$ and Γ a finite subgroup (even trivial) of $SU(m)$. We can find a function $W^{(4-2m)}_p \in C^{4,\alpha}_{4-2m}\left(M_{\boldsymbol{p}}\right)$ (actually $C^{\infty}_{loc}\left(M_{\boldsymbol{p}}\right)$)*

$$\mathbb{L}_g W^{(4-2m)}_p = -\frac{4(m-2)(m-1)\mu\left(S^{2m-1}\right)}{|\Gamma|}\delta_p + \theta^{4-2m}_p$$

with $\theta_p^{4-2m} \in C^{0,\alpha}(M)$ *and expansion*

$$W_p^{(4-2m)} = |z|^{4-2m} + \mathcal{O}\left(|z|^{6-2m}\right) . \tag{2.1}$$

Moreover, since $W_p^{(4-2m)} \in L^1(M)$ *we define*

$$\tilde{W}_p^{(4-2m)} := W_p^{(4-2m)} - \frac{1}{Vol_g(M)} \int_M W_p^{(4-2m)} d\mu_g - \sum_{i=1}^{d} \varphi_i \int_M W_p^{(4-2m)} \varphi_i d\mu_g .$$

Proof. In a small ball $B_{2r_0}(p)$ with $p \in M$ we would like to solve

$$\mathbb{L}_g u = \delta_p .$$

If we look at the Euclidean case we have that

$$\Delta^2 |z|^{4-2m} = c_m \delta_0 \qquad c_m \in \mathbb{R}$$

and so on $B_{2r_0}(p)$

$$\mathbb{L}_g |z|^{4-2m} = \mathcal{O}\left(|z|^{2-2m}\right) .$$

We will use the same technique of Proposition 2.2. Using Theorem 2.2 we first solve

$$(*_0) : \begin{cases} \Delta^2 w_0 = -2\mathbb{L}_g\left(|z|^{4-2m}\right) \\ w_0 \mid_{\partial B_\rho(p)} = 0 \\ \Delta w_0 \mid_{\partial B_\rho(p)} = 0 \end{cases}$$

and we get $w_0 \in C^{4,\alpha}_{6-2m}(B_\rho(p) \setminus \{p\})$. Now we set up a family of auxiliary problems for $k \in \mathbb{N}^+$

$$(*_k) : \begin{cases} \Delta^2 w_k = \left[2\mathbb{L}_g - \Delta^2\right](w_{k-1}) \\ w_k \mid_{\partial B_\rho(p)} = 0 \\ \Delta w_k \mid_{\partial B_\rho(p)} = 0 \end{cases}$$

and we get $w_k \in C^{4,\alpha}_{6-2m+2k}(B_\rho(p) \setminus \{p\})$. If we set

$$u_N = \sum_{k=0}^{N} (-1)^k w_k ,$$

then we have

$$\mathbb{L}_g(v_0 + u_N) = (-1)^N \left[\mathbb{L}_g - \frac{\Delta^2}{2}\right](w_{N-1})$$

and for $N = m-1$ we have that $\mathbb{L}_g(v_0 + u_{m-1}) \in C^{0,\alpha}(B_r(p))$. Let $r_0 < \rho$ then we define

$$W_p^{(4-2m)} := \left(|z|^{4-2m} + u_{m-1}\right)\chi_{r_0}$$

We now check that $W_p^{(4-2m)}$ is an approximate solution. To see it we integrate it with a test function $h \in C_0^\infty\left(B_\rho\left(p\right)\right)$ on $A_{2r_0,\varepsilon} := B_{2r_0}(p) \setminus B_\varepsilon(p)$ and then let ε tend to 0

$$
\begin{aligned}
\int_{A_{2r_0,\varepsilon}} h\mathbb{L}_g W_p^{(4-2m)} d\mu_g =& \frac{1}{2}\int_{\partial A_{2r_0,\varepsilon}} h\partial_\nu \Delta_g W_p^{(4-2m)} d\mu_g - \frac{1}{2}\int_{\partial A_{2r_0,\varepsilon}} \partial_\nu h \Delta_g W_p^{(4-2m)} d\mu_g \\
&+ \frac{1}{2}\int_{\partial A_{2r_0,\varepsilon}} \Delta_g h \partial_\nu W_p^{(4-2m)} d\mu_g - \frac{1}{2}\int_{\partial A_{2r_0,\varepsilon}} \partial_\nu \Delta_g h W_p^{(4-2m)} d\mu_g \\
&+ \frac{s_g}{2m}\int_{\partial A_{2r_0,\varepsilon}} h \partial_\nu W_p^{(4-2m)} d\mu_g - \frac{s_g}{2m}\int_{\partial A_{2r_0,\varepsilon}} \partial_\nu h W_p^{(4-2m)} d\mu_g \\
&+ 2\int_{\partial A_{2r_0,\varepsilon}} h \left(\mathrm{Ric}^0\right)^\sharp \left(\partial^\sharp W_p^{(4-2m)}\right) \lrcorner d\mu_g \\
&- 2\int_{\partial A_{2r_0,\varepsilon}} W_p^{(4-2m)} \overline{\left(\mathrm{Ric}^0\right)^\sharp \left(\partial^\sharp h\right)} \lrcorner d\mu_g \\
&+ \int_{A_{2r_0,\varepsilon}} W_p^{(2-2m)} \mathbb{L}_g h d\mu_g
\end{aligned}
$$

with

$$
\rho_g^0 = \rho_g - \frac{s_g}{2m}\omega_g \,.
$$

In $B_{2r_0}\left(p\right)$, in normal coordinates at p, we have that

$$
\begin{aligned}
\nabla_g &= \nabla_0 + \mathcal{O}\left(|z|^2\right) \,, \\
\Delta_g &= \Delta + \mathcal{O}\left(|z|^2\right) \,, \\
\left\langle \nabla_g f, \nu \right\rangle_g &= \partial_\rho f + \mathcal{O}\left(|z|^2\right) \,, \\
d\mu_g &= \left(1 + \mathcal{O}\left(|z|^2\right)\right) d\mu_0 \,.
\end{aligned}
$$

Since $W_p^{(4-2m)}$ is identically 0 on ∂B_{2r_0} we have

$$
\begin{aligned}
\int_{A_{2r_0,\varepsilon}} h\mathbb{L}_g W_p^{(4-2m)} d\mu_g =& -\frac{1}{2}\int_{\partial B_\varepsilon(p)} h\partial_\nu \Delta_g W_p^{(4-2m)} d\mu_g + \frac{1}{2}\int_{\partial B_\varepsilon(p)} \partial_\nu h \Delta_g W_p^{(4-2m)} d\mu_g \\
&- \frac{1}{2}\int_{\partial B_\varepsilon(p)} \Delta_g h \partial_\nu W_p^{(4-2m)} d\mu_g + \frac{1}{2}\int_{\partial B_\varepsilon(p)} \partial_\nu \Delta_g h W_p^{(4-2m)} d\mu_g \\
&- \frac{s_g}{2m}\int_{\partial B_\varepsilon} h \partial_\nu W_p^{(4-2m)} d\mu_g + \frac{s_g}{2m}\int_{\partial B_\varepsilon(p)} \partial_\nu h W_p^{(4-2m)} d\mu_g \\
&- 2\int_{\partial B_\varepsilon(p)} h \left(\mathrm{Ric}^0\right)^\sharp \left(\partial^\sharp W_p^{(4-2m)}\right) \lrcorner d\mu_g \\
&+ 2\int_{\partial B_\varepsilon(p)} W_p^{(4-2m)} \overline{\left(\mathrm{Ric}^0\right)^\sharp \left(\partial^\sharp h\right)} \lrcorner d\mu_g \\
&+ \int_{A_{2r_0,\varepsilon}} W_p^{(2-2m)} \mathbb{L}_g h d\mu_g \\
=& -\int_{\partial B_\varepsilon(p)} 4\left(m-2\right)\left(m-1\right) h\left(y\right) \left(|z|^{1-2m} + \mathcal{O}\left(|z|^{2-2m}\right)\right) d\mu_0 \\
&+ \int_{\partial B_\varepsilon(p)} T\left(h, \nabla h, \nabla^2 h, \nabla^3 h\right) \mathcal{O}\left(|z|^{2-2m}\right) d\mu_0 + \int_M W_p^{(4-2m)} \mathbb{L}_g h \, d\mu_g \,.
\end{aligned}
$$

With T a linear combination of derivatives up to order 3 of h. Letting ε tend to 0 we have

$$\lim_{\varepsilon\to 0}\int_{A_{2r_0,\varepsilon}} h\mathbb{L}_g W_p^{(4-2m)} d\mu_g = -4(m-2)(m-1)\frac{\mu\left(S^{2m-1}\right)}{|\Gamma|} h\left(p\right) + \int_M W_p^{(4-2m)}\mathbb{L}_g h\, d\mu_g$$

and if $h \in \ker\left(\mathbb{L}_g\right)$ we have

$$\lim_{\varepsilon\to 0}\int_{A_{2r_0,\varepsilon}} h\mathbb{L}_g W_p^{(4-2m)} d\mu_g = -4(m-2)(m-1)\frac{\mu\left(S^{2m-1}\right)}{|\Gamma|} h\left(p\right) .$$

Concluding we have that on $B_{2r_0}\left(p\right)$

$$\mathbb{L}_g W_p^{(4-2m)} = -\frac{4(m-2)(m-1)\mu\left(S^{2m-1}\right)}{|\Gamma|}\delta_p + (-1)^{m-1}\chi_{r_0}\widetilde{\mathbb{L}}_g\left(w_{m-1}\right) + T\left(\chi_{r_0}, v_0+u_{m-1}\right)$$

with $T\left(,\right)$ a linear espression involving derivatives of order at least 1 of χ_{r_0} and at most 3 of v_0+u_{m-1} so we have proved that $W_p^{(4-2m)}$ is an approximate solution of $\mathbb{L}_g\left(u\right)=0$. □

In this new setting we need to find out what kind of distribution give rise an approximate solution blowing up at rate $2-2m$. Also Szekelyhidi in [Szé13] needs to construct this kind of function and he does it with a technique very similar to ours.

Proposition 2.4. *Let $p \in M$, $m \geq 3$ and Γ a finite subgroup (even trivial) of $SU(m)$. We can find a function $W_p^{2-2m} \in C^{4,\alpha}_{2-2m}\left(M\right)$ (actually $C^{\infty}_{loc}\left(M\right)$) s.t.*

$$\mathbb{L}_g W_p^{(2-2m)} = \frac{(m-1)\mu\left(S^{2m-1}\right)}{|\Gamma|}\left(\Delta_g\delta_p + \frac{s_g}{m}\delta_p + \frac{s_g\left(m-1\right)^2}{m\left(m+1\right)}\delta_p\right) + \theta_p^{2-2m}$$

with $\theta_p^{2-2m} \in C^{0,\alpha}\left(M\right)$ (actually $\theta_p^{2-2m} \in C^{\infty}\left(M\right)$) and near p has the following expansion

$$W_p^{(2-2m)} = |z|^{2-2m} + \left(\tilde{\phi}_2+\tilde{\phi}_4\right)|z|^{4-2m} + |z|^{5-2m}\sum_{h=1}^{K}\tilde{\phi}_{2h+1} + \mathcal{O}\left(|z|^{6-2m}\right) . \tag{2.2}$$

Since $W_p^{(2-2m)} \in L^1\left(M\right)$ we define

$$\tilde{W}_p^{(2-2m)} := W_p^{(2-2m)} - \frac{1}{Vol_g\left(M\right)}\int_M W_p^{(2-2m)} d\mu_g - \sum_{i=1}^{d}\varphi_i\int_M W_p^{(2-2m)}\varphi_i d\mu_g .$$

Proof. We'd like to solve the equation

$$\mathbb{L}_g v = \Delta_g\delta_p$$

and thinking about the euclidean case we start looking at the function

$$|z|^{2-2m} .$$

The error we commit is

$$
\begin{aligned}
\mathbb{L}_g|z|^{2-2m} =& \widetilde{\mathbb{L}}_g|z|^{2-2m} \\
=& -4\mathrm{tr}\left(\partial\overline{\partial}|z|^{2-2m}\cdot\partial\overline{\partial}\Delta\psi_g\right)-4\mathrm{tr}\left(\partial\overline{\partial}\psi_g\cdot\partial\overline{\partial}\Delta|z|^{2-2m}\right)-4\Delta\mathrm{tr}\left(\partial\overline{\partial}\psi_g\cdot\partial\overline{\partial}|z|^{2-2m}\right) \\
&+\mathcal{O}\left(|z|^{2-2m}\right) \\
=& -4\mathrm{tr}\left(\partial\overline{\partial}|z|^{2-2m}\cdot\partial\overline{\partial}\Delta p_4\right)-4\mathrm{tr}\left(\partial\overline{\partial}p_4\cdot\partial\overline{\partial}\Delta|z|^{2-2m}\right)-4\Delta\mathrm{tr}\left(\partial\overline{\partial}p_4\cdot\partial\overline{\partial}|z|^{2-2m}\right) \\
&+|z|^{1-2m}\sum_{h=1}^{K}\phi_{2h+1}+\mathcal{O}\left(|z|^{2-2m}\right) \\
=& -4\mathrm{tr}\left(\partial\overline{\partial}|z|^{2-2m}\cdot\partial\overline{\partial}\Delta p_4\right)-4\Delta\mathrm{tr}\left(\partial\overline{\partial}p_4\cdot\partial\overline{\partial}|z|^{2-2m}\right)+\mathcal{O}\left(|z|^{2-2m}\right) \\
=&(m-1)\frac{\Delta^2 p_4}{|z|^{2m}}-4(m-1)m\frac{\Delta p_4}{|z|^{2m+2}}-(m-1)\Delta\left(\frac{\Delta p_4}{|z|^{2m}}\right) \\
&+16(m-1)m\Delta\left(\frac{p_4}{|z|^{2m+2}}\right) \\
&+|z|^{1-2m}\sum_{h=1}^{K}\tilde{\phi}_{2h+1}+\mathcal{O}\left(|z|^{2-2m}\right) \\
=&\frac{1}{|z|^{2m}}\tilde{\phi}_2+\frac{1}{|z|^{2m}}\tilde{\phi}_4+|z|^{1-2m}\sum_{h=1}^{K}\tilde{\phi}_{2h+1}+\mathcal{O}\left(|z|^{2-2m}\right)\,.
\end{aligned}
$$

In the above computations we used formulas (1.4) and (1.7). We introduce corrections

$$v_2=\frac{c_2}{|z|^{2m-4}}\tilde{\phi}_2\,,$$

$$v_4=\frac{c_4}{|z|^{2m-4}}\tilde{\phi}_4\,,$$

$$\hat{v}=\frac{1}{|z|^{2m-5}}\sum_{h=1}^{K}c_{2h+1}\tilde{\phi}_{2h+1}$$

and so we have

$$\mathbb{L}_g\left(v_0-v_2-v_4-\hat{v}\right)=\mathcal{O}(|z|^{2-2m})\,.$$

Now we argue as in Proposition 2.2. Using Theorem 2.2 we first solve

$$(*_0):\begin{cases}\Delta^2 w_0=2\mathbb{L}_g\left(v_0-v_2-v_4-\hat{v}\right)\\ w_0\mid_{\partial B_{\bar{r}}(p)}=0\\ \Delta w_0\mid_{\partial B_{\bar{r}}(p)}=0\end{cases}$$

and we get $w_0\in C^{4,\alpha}_{6-2m}\left(B_{\bar{r}}\left(p\right)\setminus\{p\}\right)$. Now we set up a family of auxiliary problems for $k\in\mathbb{N}^+$

$$(*_k):\begin{cases}\Delta^2 w_k=\left[2\mathbb{L}_g-\Delta^2\right](w_{k-1})\\ w_k\mid_{\partial B_{\bar{r}}(p)}=0\\ \Delta w_k\mid_{\partial B_{\bar{r}}(p)}=0\end{cases}$$

and we get $w_k \in C^{4,\alpha}_{6-2m+2k}\left(B_{\bar{r}}\left(p\right) \setminus \{p\}\right)$. If we set

$$w_N = \sum_{k=0}^{N} \left(-1\right)^k w_k \,,$$

then we have

$$\mathbb{L}_g \left(v_0 - v_2 - v_4 - \hat{v} + w_N\right) = \left(-1\right)^N \left[\mathbb{L}_g - \frac{\Delta^2}{2}\right] \left(w_{N-1}\right)$$

and for $N = m-1$ we have that $\mathbb{L}_g \left(|z|^{2-2m} - v_2 - v_4 - \hat{v} + w_{m-1}\right) \in C^{0,\alpha}\left(B_{\bar{r}}\left(p\right)\right)$. Let $r_0 < \bar{r}$, we consider now the function

$$W_p^{(2-2m)} := \left(|z|^{2-2m} - v_2 - v_4 - \hat{v} + u_{m-1}\right) \chi_{r_0} \,.$$

What is $\mathbb{L}_g W_p^{(2-2m)}$ in distributional sense? To see it we integrate it with a test function $h \in C^\infty(M)$ on $A_{r_0,\varepsilon} := B_{r_0}(p) \setminus B_\varepsilon(p)$ and then let ε tend to 0

We recall that

$$\mathbb{L}_g f = \frac{\Delta_g^2}{2} f + \frac{s_g}{2m} \Delta_g f + 2 \left\langle \rho_g^0, i\partial\overline{\partial} f \right\rangle_g$$

with

$$\rho_g = i \mathrm{Ric}_{i\bar{j}} dz^i \wedge \overline{dz^j} \qquad \rho_g^0 = \rho_g - \frac{s_g}{2m} \omega_g \,.$$

Integrating with a test function h we have

$$\begin{aligned} \int_{A_{2r_0,\varepsilon}} h \mathbb{L}_g W_p^{(2-2m)} d\mu_g =& \frac{1}{2} \int_{A_{2r_0,\varepsilon}} h \left[\Delta_g^2 W_p^{(2-2m)} + \frac{s_g}{m} \Delta_g W_p^{(2-2m)}\right] d\mu_g \\ &+ 2 \int_{A_{2r_0,\varepsilon}} h \left\langle \rho_g^0, i\partial\overline{\partial} W_p^{(2-2m)} \right\rangle_g d\mu_g \,. \end{aligned}$$

First we perform integration by parts with the easiest part of $\mathbb{L}_g$

$$\begin{aligned} \frac{1}{2} \int_{A_{2r_0,\varepsilon}} h \left[\Delta_g^2 W_p^{(2-2m)} + \frac{s_g}{m} \Delta_g W_p^{(2-2m)}\right] d\mu_g =& \frac{1}{2} \int_{\partial A_{2r_0,\varepsilon}} h \partial_\nu \Delta_g W_p^{(2-2m)} d\mu_g \\ &- \frac{1}{2} \int_{\partial A_{2r_0,\varepsilon}} \partial_\nu h \Delta_g W_p^{(2-2m)} d\mu_g \\ &+ \frac{1}{2} \int_{\partial A_{2r_0,\varepsilon}} \Delta_g h \partial_\nu W_p^{(2-2m)} d\mu_g \\ &- \frac{1}{2} \int_{\partial A_{2r_0,\varepsilon}} \partial_\nu \Delta_g h W_p^{(2-2m)} d\mu_g \\ &+ \frac{s_g}{2m} \int_{\partial A_{2r_0,\varepsilon}} h \partial_\nu W_p^{(2-2m)} d\mu_g \\ &- \frac{s_g}{2m} \int_{\partial A_{2r_0,\varepsilon}} \partial_\nu h W_p^{(2-2m)} d\mu_g \\ &+ \frac{1}{2} \int_{A_{2r_0,\varepsilon}} W_p^{(2-2m)} \left[\Delta_g^2 h + \frac{s_g}{m} \Delta_g h\right] d\mu_g \,. \end{aligned}$$

In this case more care than in Proposition 1.1 is needed with boundary terms. Using Lemma 1.1 we compute

$$\begin{aligned}
\Delta_g |z|^{2-2m} &= 2g^{i\bar{j}}\partial_j\overline{\partial}_i |z|^{2-2m} \\
&= 4\left(\delta_{i\bar{j}} - 2\partial\overline{\partial}p_4 + \mathcal{O}\left(|z|^3\right)\right)\partial_j\overline{\partial}_i |z|^{2-2m} \\
&= -8\mathrm{tr}\left(\partial\overline{\partial}p_4 \cdot \partial\overline{\partial}|z|^{2-2m}\right) + \mathcal{O}\left(|z|^{3-2m}\right) \\
&= \frac{2(m-1)}{|z|^{2m}}\Delta p_4 - \frac{32(m-1)m}{|z|^{2m}}p_4 + \mathcal{O}\left(|z|^{3-2m}\right) \\
&= \frac{s_g(m-1)^2}{m(m+1)|z|^{2m-2}} + \frac{1}{|z|^{2m-2}}\tilde{\phi}_2 + \frac{1}{|z|^{2m-2}}\tilde{\phi}_4 + \mathcal{O}\left(|z|^{3-2m}\right)
\end{aligned}$$

and then

$$\begin{aligned}
\partial_\nu\Delta_g |z|^{2-2m} &= \partial_\rho\Delta_g |z|^{2-2m} + \mathcal{O}\left(|z|^{2-2m}\right) \\
&= -\frac{2(m-1)}{|z|^{2m-1}}\left[\frac{s_g(m-1)^2}{m(m+1)} + \tilde{\phi}_2 + \tilde{\phi}_4\right] + \mathcal{O}\left(|z|^{2-2m}\right) \\
&= \frac{1}{|z|^{2m-1}}\left[-\frac{2s_g(m-1)^3}{m(m+1)} + \tilde{\phi}_2 + \tilde{\phi}_4\right] + \mathcal{O}\left(|z|^{2-2m}\right).
\end{aligned}$$

So now we have

$$\begin{aligned}
\int_M h\left[\frac{\Delta_g^2}{2} + \frac{s_g}{2m}\Delta_g\right]\left(W_p^{(2-2m)}\right)d\mu_g &= \lim_{\varepsilon\to 0}\frac{1}{2}\int_{A_{2r_0,\varepsilon}} h\left[\Delta_g^2 W_p^{(2-2m)} + \frac{s_g}{m}\Delta_g W_p^{(2-2m)}\right]d\mu_g \\
&= \frac{(m-1)\mu\left(S^{2m-1}\right)}{|\Gamma|}\left(\Delta_g h(p) + \frac{s_g\left(m^2-m+2\right)}{m(m+1)}h(p)\right) \\
&\quad + \frac{1}{2}\int_M W_p^{(2-2m)}\left[\Delta_g^2 h + \frac{s_g}{m}\Delta_g h\right]d\mu_g\,.
\end{aligned}$$

Now we deal with the term contanining Ric^0.

$$
\begin{aligned}
\int_{A_{2r_0,\varepsilon}} h \left\langle \rho_g^0, i\partial\overline{\partial} W_p^{(2-2m)} \right\rangle_g d\mu_g =& \int_{A_{2r_0,\varepsilon}} h g^{i\bar{b}} \mathrm{Ric}^0_{i\bar{j}} g^{a\bar{j}} \nabla_a \nabla_{\bar{b}} W_p^{(2-2m)} d\mu_g \\
=& \int_{A_{2r_0,\varepsilon}} \nabla_a \left[h g^{i\bar{b}} \mathrm{Ric}^0_{i\bar{j}} g^{a\bar{j}} \nabla_{\bar{b}} W_p^{(2-2m)} \right] d\mu_g \\
&- \int_{A_{2r_0,\varepsilon}} \nabla^{\bar{j}} \left(\mathrm{Ric}^0_{i\bar{j}} \right) h g^{i\bar{b}} \nabla_{\bar{b}} W_p^{(2-2m)} d\mu_g \\
&- \int_{A_{2r_0,\varepsilon}} \mathrm{Ric}^0_{i\bar{j}} \nabla^{\bar{j}} h \nabla^i W_p^{(2-2m)} d\mu_g \\
=& \int_{A_{2r_0,\varepsilon}} \nabla_a \left[h g^{i\bar{b}} \mathrm{Ric}^0_{i\bar{j}} g^{a\bar{j}} \nabla_{\bar{b}} W_p^{(2-2m)} \right] d\mu_g \\
&- \int_{A_{2r_0,\varepsilon}} \nabla^i \left[\mathrm{Ric}^0_{i\bar{j}} \nabla^{\bar{j}} h W_p^{(2-2m)} \right] d\mu_g \\
&- \int_{A_{2r_0,\varepsilon}} \nabla^i \left(\mathrm{Ric}^0_{i\bar{j}} \right) \nabla^{\bar{j}} h W_p^{(2-2m)} d\mu_g \\
&+ \int_{A_{2r_0,\varepsilon}} W_p^{(2-2m)} \left\langle \rho_g^0, i\partial\overline{\partial} h \right\rangle_g d\mu_g \\
=& \int_{A_{2r_0,\varepsilon}} \nabla_a \left[h \left(\mathrm{Ric}^0 \right)_i^a \nabla^i W_p^{(2-2m)} \right] d\mu_g \\
&- \int_{A_{2r_0,\varepsilon}} \nabla^i \left[\mathrm{Ric}^0_{i\bar{j}} \nabla^{\bar{j}} h W_p^{(2-2m)} \right] d\mu_g \\
&+ \int_{A_{2r_0,\varepsilon}} W_p^{(2-2m)} \left\langle \rho_g^0, i\partial\overline{\partial} h \right\rangle_g d\mu_g \\
=& \int_{\partial A_{2r_0,\varepsilon}} h \left(\mathrm{Ric}^0 \right)^\sharp \left(\partial^\sharp W_p^{(2-2m)} \right) \lrcorner d\mu_g \\
&- \int_{\partial A_{2r_0,\varepsilon}} W_p^{(2-2m)} \overline{\left(\mathrm{Ric}^0 \right)^\sharp \left(\partial^\sharp h \right)} \lrcorner d\mu_g \\
&+ \int_{A_{2r_0,\varepsilon}} W_p^{(2-2m)} \left\langle \rho_g^0, i\partial\overline{\partial} h \right\rangle_g d\mu_g \,.
\end{aligned}
$$

We recall the expansions

$$
\begin{aligned}
\left(\mathrm{Ric}^0 \right)_i^j &= \left(\mu_i \left(p \right) - \frac{s_g}{2m} \right) \delta_i^j + \mathcal{O} \left(|z| \right) \,, \\
\partial^\sharp W_p^{(2-2m)} &= \sum_{i=1}^m \left(1 - m \right) \frac{z^i}{|z|^{2m}} \partial_i + \mathcal{O} \left(|z|^{2-2m} \right) \\
d\mu_g &= d\mu_0 + \mathcal{O} \left(|z|^2 \right) \,.
\end{aligned}
$$

So we get

$$
\left(\mathrm{Ric}^0 \right)^\sharp \left(\partial^\sharp W_p^{(2-2m)} \right) \lrcorner d\mu_g = \left(1 - m \right) \sum_{i=1}^m \left(\mu_i \left(p \right) - \frac{s_g}{2m} \right) z^i \left(\partial_i \lrcorner d\mu_0 \right) + \mathcal{O} \left(|z|^1 \right) \,.
$$

We set

$$I_i = \int_{\partial A_{2r_0,\varepsilon}} z^i \left(\partial_i \lrcorner d\mu_0\right) .$$

Clearly $|I_i| < +\infty$ and by symmmetry we have that

$$I_k = I_l = I \qquad 1 \leq k, l \leq m$$

so

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_{A_{2r_0,\varepsilon}} h \left\langle \rho_g^0, i\partial\overline{\partial} W_p^{(2-2m)} \right\rangle_g d\mu_g &= (m-1)\, h(p)\, I \frac{\mu\left(S^{2m-1}\right)}{|\Gamma|} \sum_{j=1}^{m} \left(\mu_j(p) - \frac{s_g}{2m}\right) \\ &\quad + \int_{A_{2r_0,\varepsilon}} W_p^{(2-2m)} \left\langle \rho_g^0, i\partial\overline{\partial} h \right\rangle_g d\mu_g \\ &= \int_M W_p^{(2-2m)} \left\langle \rho_g^0, i\partial\overline{\partial} h \right\rangle_g d\mu_g .\end{aligned}$$

Putting all the piecies together we have

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_{\partial A_{2r_0,\varepsilon}} \mathbb{L}_g W_p^{(2-2m)} h d\mu_g =& \frac{(m-1)\, \mu\left(S^{2m-1}\right)}{|\Gamma|} \left[\frac{s_g (m-1)^2}{m(m+1)} h(p) + \Delta_g h(p) + \frac{s_g}{m} h(p) \right] \\ &+ \int_{B_{2r_0}} W_p^{(2-2m)} \mathbb{L}_g h d\mu_g .\end{aligned}$$

Concluding, we have that

$$\begin{aligned}\mathbb{L}_g W_p^{(2-2m)} =& \frac{(m-1)\, \mu\left(S^{2m-1}\right)}{|\Gamma|} \left[\frac{s_g (m-1)^2}{m(m+1)} \delta_p + \Delta_g \delta_p + \frac{s_g}{m} \delta_p \right] \\ &+ (-1)^{m-1} \chi_{r_0} \left[\mathbb{L}_g - \frac{\Delta^2}{2} \right] (w_{m-1}) + T\left(\chi_{r_0}, |z|^{2-2m} - v_2 - v_4 + w_{m-1}\right)\end{aligned}$$

with $T(,)$ a linear espression involving derivatives of order at least 1 of χ_{r_0} and at most 3 of $|z|^{2-2m} - v_2 - v_4 + w_{m-1}$ so we have proved that $W_p^{(2-2m)}$ is an approximate solution of $\mathbb{L}_g(u) = 0$ $\square$

We now prove an existence result with estimates for equations with distribution data. The following proposition is a more general version of a result that Arezzo and Pacard used in [AP09].

Proposition 2.5. *Let (M, g) be a compact cscK orbifold with isolated singularities,*

$$p_1, \ldots, p_N, q_1, \ldots q_{N'} \in M$$

points on M and

$$\langle 1, \varphi_1, \ldots, \varphi_d \rangle = \ker(\mathbb{L}_g)$$

a L^2-orthonormal base of $\ker(\mathbb{L}_g)$. *Let $c_0, \ldots, c_d \in \mathbb{R}$, then there exist a distribution*

$$H[\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{d}] \in \mathcal{D}'(M)$$

such that

$$\mathbb{L}_g H\left[\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{d}\right] + a_0 = \sum_{j=1}^{N} a_j \delta_{p_j} + \sum_{k=1}^{N'} \left(b_k \Delta_g \delta_{q_k} + d_k \delta_{q_k}\right) + \sum_{l=1}^{d} c_l \varphi_l + c_0$$

if

$$c_0\, Vol_g(M) - a_0\, Vol_g(M) + \sum_{j=1}^{N} a_j + \sum_{k=1}^{N'} d_k = 0 \tag{2.3}$$

$$\sum_{j=1}^{N} a_j \varphi_l(p_j) + \sum_{k=1}^{N'} \left(b_k \Delta_g \varphi_l(q_k) + \sum_{k=1}^{N'} d_k \varphi_l\left(q_k\right) \right) + c_l = 0 \quad 1 \leq l \leq d \tag{2.4}$$

Moreover

$$H\left[\boldsymbol{a}, \boldsymbol{c}, \boldsymbol{b}, \boldsymbol{d}\right] \in C^{\infty}_{loc}\left(M \setminus \{p_1, \ldots, p_N, q_1, \ldots q_{N'}\}\right) \cap C^{4,\alpha}_{4-2m,2-2m}\left(M \setminus \{p_1, \ldots, p_N, q_1, \ldots q_{N'}\}\right).$$

Proof. We call

$$\mu = \sum_{j=1}^{N} a_j \delta_{p_j} + \sum_{k=1}^{N'} \left(b_k \Delta_g \delta_{q_k} + d_k \delta_{q_k}\right) + \sum_{l=1}^{d} c_l \varphi_l + c_0 - a_0 \,.$$

Conditions (2.3) and (2.4) translate to the fact that

$$\mu\left[\varphi\right] = 0 \qquad \varphi \in \ker\left(\mathbb{L}_g\right)$$

and we define a distribution T (our candidate $H\left[\mathbf{a}, \mathbf{c}, \mathbf{b}, \mathbf{d}\right]$) as

$$T\left[\phi\right] = \mu\left[S_{\mathbb{L}_g}\left(\phi^{\perp}\right)\right] \qquad \phi \in C^{\infty}\left(M\right)$$

with

$$\phi^{\perp} = \phi - \frac{1}{\mathrm{Vol}_g\left(M\right)} \int_M \phi d\mu_g - \sum_{i=1}^{d} \varphi_i \int_M \phi \varphi_i d\mu_g \,.$$

We now show that T is indeed a solution of

$$\mathbb{L}_g T = \mu \,.$$

Indeed we have

$$\begin{aligned} \mathbb{L}_g\left(T\right)\left[\phi\right] =& T\left[\mathbb{L}_g^*\left(\phi\right)\right] \\ =& T\left[\mathbb{L}_g\left(\phi\right)\right] \\ =& T\left[\mathbb{L}_g\left(\phi^{\perp}\right)\right] \\ =& \mu\left[S_{\mathbb{L}_g}\left(\mathbb{L}_g\left(\phi^{\perp}\right)^{\perp}\right)\right] \\ =& \mu\left[S_{\mathbb{L}_g} \circ \mathbb{L}_g\left(\phi^{\perp}\right)\right] \\ =& \mu\left[\phi^{\perp}\right] \\ =& \mu\left[\phi\right] \,. \end{aligned}$$

In the last equation above we used conditions (2.3) and (2.4), that are the general case of the balancing condition (1.10) of Theorem 1.7. We now prove the estimates. We consider the distribution

$$T^{'} = T - L_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\left(\tilde{W}^{4-2m}_{\mathbf{p}}, \tilde{W}^{4-2m}_{\mathbf{q}}, \tilde{W}^{2-2m}_{\mathbf{q}}\right)$$

with $L_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$ a linear function with coefficients depending on only on $m, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. By construction we have that

$$\mathbb{L}_g\left(T^{'}\right) = L^{'}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\left(\tilde{\theta}^{4-2m}_{\mathbf{p}}, \tilde{\theta}^{4-2m}_{\mathbf{q}}, \tilde{\theta}^{2-2m}_{\mathbf{q}}\right)$$

with $L^{'}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}$ a linear function with coefficients depending on only on $m, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Since the right hand side belongs to $C^{0,\alpha}(M)$ (actually $C^{\infty}(M)$) by Theorem A.14 we have that $T^{'} \in C^{4,\alpha}(M)$ (actually $T^{'} \in C^{\infty}(M)$) and moreover since $T^{'}$ is orthogonal to $\ker(\mathbb{L}_g)$ we have the estimate

$$\left\|T^{'}\right\|_{C^{4,\alpha}(M)} \leq C \left\|L^{'}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}}\left(\tilde{\theta}^{4-2m}_{\mathbf{p}}, \tilde{\theta}^{4-2m}_{\mathbf{q}}, \tilde{\theta}^{2-2m}_{\mathbf{q}}\right)\right\|_{C^{0,\alpha}(M)}$$

and so we have

$$\left\|T^{'}\right\|_{C^{4,\alpha}(M)} \leq C\left(|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}|\right)$$

and the proposition follows. □

Remark 2.2. In the sequel we will use a specialized version of the above proposition, indeed we will put some restrictions on coefficients of equations 2.3 and 2.4.

2.2 Analysis on ALE manifolds

In this section we focus our attention to the linear analysis on *ALE* spaces. Many of the results of this section are well known but for clearness we give the proofs.

2.2.1 The equation $\mathbb{L}_\eta u = f$

We start this section with an innocent observation that will be very important for the forthcoming calculations.

Lemma 2.2. *Let* (X, η) *be a Ricci flat ALE Kähler space coming from a crepant resolution of* $\mathbb{C}^m/\Gamma$ *with* $\Gamma \triangleleft SU(m)$

$$\pi : X \to \mathbb{C}^m/\Gamma\,,$$

then on $X \setminus \pi^{-1}(0)$ *we have*

$$d\mu_\eta = d\mu_0\,,$$

and for $\rho > 0$

$$Vol_\eta\left(X_\rho\right) = \frac{\mu\left(S^{2m-1}\right)}{2m\,|\Gamma|}\rho^{2m}\,.$$

Proof. Let $\pi_\Gamma : \mathbb{C}^m \to \mathbb{C}^m/\Gamma$ the canonical holomorphic quotient map, since

$$\mathrm{Ric}\,(\eta) = 0\,,$$

on $\left(\mathbb{C}^m \setminus B_\rho\right)/\Gamma$ we have

$$i\partial\overline{\partial}\left[\log\left(\det\left(\left(\pi_\Gamma\right)^*\left(\pi^{-1}\right)^*\eta\right)\right)\right] = 0\,.$$

We want to prove that on

$$\mathbb{C}^m \setminus \{0\}$$

$$\det\left(\left(\pi_\Gamma\right)^* \left(\pi^{-1}\right)^* \eta\right) \equiv C \qquad C \in \mathbb{R}\,.$$

First of all on $\mathbb{C}^m \setminus B_\rho$ we have

$$\left(\pi_\Gamma\right)^* \left(\pi^{-1}\right)^* \eta = \frac{\delta_{i\bar{j}}}{2} - c_\Gamma \partial\overline{\partial} \left|x\right|^{2-2m} + \partial\overline{\partial}\mathcal{O}\left(\left|x\right|^{2-2m}\right)$$

and so

$$\begin{aligned}
\det\left(\left(\pi_\Gamma\right)^* \left(\pi^{-1}\right)^* \eta\right) &= \det\left(\frac{\delta_{i\bar{j}}}{2} + c\partial\overline{\partial} \left|x\right|^{2-2m} + \partial\overline{\partial}\mathcal{O}\left(\left|x\right|^{-2m}\right)\right)\\
&= \frac{1}{2^m}\det\left(\delta_{i\bar{j}} + 2c_\Gamma \partial\overline{\partial} \left|x\right|^{2-2m} + 2\partial\overline{\partial}\mathcal{O}\left(\left|x\right|^{-2m}\right)\right)\\
&= \frac{1}{2^m}\left[1 + \sum_{k=1}^{m} 2^k \sigma_k \left(c_\Gamma \partial\overline{\partial} \left|x\right|^{2-2m} + \partial\overline{\partial}\mathcal{O}\left(\left|x\right|^{-2m}\right)\right)\right]\\
&= \frac{1}{2^m}\left[1 + \frac{1}{2}\Delta\mathcal{O}\left(\left|x\right|^{-2m}\right) + \sum_{k=2}^{m} 2^k \sigma_k \left(c_\Gamma \partial\overline{\partial} \left|x\right|^{2-2m} + \partial\overline{\partial}\mathcal{O}\left(\left|x\right|^{-2m}\right)\right)\right]\\
&= \frac{1}{2^m}\left[1 + \mathcal{O}\left(\left|x\right|^{-2-2m}\right)\right]\,.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
\log\left(\det\left(\left(\pi_\Gamma\right)^* \left(\pi^{-1}\right)^* \eta\right)\right) &= \log\left(\frac{1}{2^m}\left[1 + \mathcal{O}\left(\left|x\right|^{-2m}\right)\right]\right)\\
&= -m\log\left(2\right) + \sum_{n=1}^{+\infty} \left(-1\right)^{n+1} \frac{\mathcal{O}\left(\left|x\right|^{-2-2m}\right)^n}{n}\\
&= -m\log\left(2\right) + \mathcal{O}\left(\left|x\right|^{-2-2m}\right)\,.
\end{aligned}$$

On $\mathbb{C}^m \setminus B_\rho$ we have

$$\begin{aligned}
i\partial\overline{\partial}\log\left(\det\left(\left(\pi_\Gamma\right)^* \left(\pi^{-1}\right)^* \eta\right)\right) &= -i\overline{\partial}\partial\log\left(\det\left(\left(\pi_\Gamma\right)^* \left(\pi^{-1}\right)^* \eta\right)\right)\\
&= -id\left(\partial\log\left(\det\left(\left(\pi_\Gamma\right)^* \left(\pi^{-1}\right)^* \eta\right)\right)\right)\,,
\end{aligned}$$

so

$$\partial\log\left(\det\left(\left(\pi_\Gamma\right)^* \left(\pi^{-1}\right)^* \eta\right)\right) \in H^1\left(\mathbb{C}^m \setminus B_\rho, \mathbb{C}\right)$$

but $H^1\left(\mathbb{C}^m \setminus B_\rho, \mathbb{C}\right) = 0$ and there exists $h_1 \in C^1\left(\mathbb{C}^m \setminus B_\rho, \mathbb{C}\right)$ such that

$$\partial\log\left(\det\left(\left(\pi_\Gamma\right)^* \left(\pi^{-1}\right)^* \eta\right)\right) = dh_1 = \partial h_1 + \overline{\partial} h_1 \qquad \Rightarrow \qquad \overline{\partial} h_1 = 0\,.$$

Analogously, we have

$$i\partial\overline{\partial}\left[\log\left(\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)\right)-h_1\right]=-id\left[\overline{\partial}\log\left(\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)\right)-\overline{\partial}h_1\right]$$

so

$$\overline{\partial}\left[\log\left(\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)\right)-h_1\right]\in H^1\left(\mathbb{C}^m\setminus B_\rho,\mathbb{C}\right)$$

and then exists $h_2\in C^1\left(\mathbb{C}^m\setminus B_\rho,\mathbb{C}\right)$ such that

$$\overline{\partial}\left[\log\left(\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)\right)-h_1\right]=dh_2=\partial h_2+\overline{\partial}h_2\qquad\Rightarrow\qquad\partial h_2=0\,.$$

Summing up all these data we have

$$\begin{aligned}d\left[\log\left(\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)\right)-h_1-h_2\right]=&\partial\left[\log\left(\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)\right)-h_1\right]\\&+\overline{\partial}\left[\log\left(\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)\right)-h_2\right]\\=&0\,.\end{aligned}$$

We conclude that on $\mathbb{C}^m\setminus B_\rho$

$$\log\left(\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)\right)=h_1+h_2+K\qquad K\in\mathbb{R}\qquad\mathfrak{Im}h_2=-\mathfrak{Im}h_1$$

moreover we note that $h_1,\overline{h_2}$ are holomorphic on $\mathbb{C}^m\setminus B_\rho$ and by Hartogs extension theorem they are extendable to functions H_1,H_2 holomorphic on $\mathbb{C}^m$. Since H_1,H_2 are holomorphic, their real and imaginary parts are harmonic with respect to the euclidean metric on $\mathbb{C}^m$ and by assumptions on η we have on $\mathbb{C}^m\setminus B_\rho$

$$\mathfrak{Re}H_1+\mathfrak{Re}H_2+K=-m\log(2)+\mathcal{O}\left(|x|^{-2-2m}\right)\,.$$

Since $\mathfrak{Re}H_1+\mathfrak{Re}H_2+K$ is harmonic and bounded, Liouville theorem implies it is constant, so

$$\log\left(\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)\right)=C\Rightarrow\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)=\frac{1}{2^m}$$

We have the following relations

$$\begin{aligned}\frac{1}{m!}(\pi_\Gamma)^*\left[\left(\pi^{-1}\right)^*\eta\right]^{\wedge m}&=\frac{1}{m!}\left[(\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right]^{\wedge m}\\&=\frac{1}{m!}\det\left((\pi_\Gamma)^*\left(\pi^{-1}\right)^*\eta\right)dx^1\wedge\overline{dx^1}\wedge\ldots\wedge dx^m\wedge\overline{dx^m}\\&=d\mu_0\,.\end{aligned}$$

So we can compute

$$\begin{aligned}\mathrm{Vol}_\eta\left(X_\rho\right)&=\int_{X_\rho}d\mu_\eta\\&=\int_{X_\rho\setminus\pi^{-1}(0)}d\mu_\eta\\&=\int_{B_\rho/\Gamma\setminus\{0\}}d\mu_{(\pi^{-1})^*\eta}\\&=\frac{\mu\left(S^{2m-1}\right)}{2m\,|\Gamma|}\rho^{2m}\end{aligned}$$

and the lemma follows. □

Let (X, η) be an *ALE* Kähler space, we define in this setting too weighted Hölder spaces and weighted Sobolev spaces.

Definition 2.4. Let (X, η) be an *ALE* Kähler space, let

$$X_{R_0} = \pi^{-1}\left(B_{R_0}\right)$$

with π the canonical surjection to $\mathbb{C}^m/\Gamma$. Let $\delta \in \mathbb{R}$, $\gamma \in C^{\infty}_{loc}(X)$ defined as

$$\gamma(p) := \chi(p) + (1 - \chi(p))\,|x(p)| \qquad p \in X$$

with χ a smooth cutoff function identically 1 on X_{R_0} and identically 0 on $X \setminus X_{2R_0}$. The weighted Sobolev space $W^{k,2}_{\delta}(X)$ is the set of functions $f \in L^1_{loc}(X)$ such that

$$\|f\|_{W^{k,2}_{\delta}(X)} := \sqrt{\sum_{j=0}^{k} \int_X |\gamma^{-\delta+j} \nabla^j f|^2_{\eta}\, d\mu_\eta} < +\infty\,.$$

Definition 2.5. Let (X, η) be an *ALE* Kähler space, let

$$X_{R_0} = \pi^{-1}\left(B_{R_0}\right)$$

with π the canonical surjection to $\mathbb{C}^m/\Gamma$. Let $\delta \in \mathbb{R}$, $\alpha \in (0,1)$, the weighted Hölder space $C^{k,\alpha}_{\delta}(X)$ is the set of functions $f \in C^{k,\alpha}_{loc}(X)$ s.t.

$$\|f\|_{C^{k,\alpha}_{\delta}(X)} := \|f\|_{C^{k,\alpha}\left(X_{R_0}\right)} + \sup_{R \geq R_0} R^{-\delta} \|f(R\cdot)\|_{C^{k,\alpha}\left(A^{\Gamma}_1\right)} < +\infty\,.$$

Remark 2.3. It is easy to see that the space $C^{k,\alpha}_{\delta}(X)$ is a separable Banach space and $W^{k,2}_{\delta}(X)$ is a separable Hilbert space. We also note that we have the inclusion

$$C^{k,\alpha}_{\delta}(X) \subseteq W^{k,2}_{\delta'+m}(X) \qquad \delta' > \delta\,.$$

For *ALE* Kähler spaces holds true a result analogous to Proposition 2.2.

Proposition 2.6. *Let (X, η) an ALE Kähler space that is scalar flat. Let $\delta \in (0,1)$ then*

$$\mathbb{L}_\eta : C^{4,\alpha}_{4-2m+\delta}(X) \longrightarrow C^{0,\alpha}_{-2m+\delta}(X)$$

is surjective and the inverse is bounded. Let $\delta \in \mathbb{R}$ with

$$\delta \neq l + m, 4 - m - l \qquad l \in \mathbb{N}\,.$$

We define the operator $\mathcal{L}_\delta$ between weighted Sobolev spaces

$$\mathcal{L}_\delta := \mathbb{L}_\eta : W^{4,2}_{\delta}(X) \to L^2_{\delta-4}(X)\,.$$

Then its adjoint operator with respect to the bilinear pairing

$$< \cdot, \cdot >_\eta :\, L^2_{\delta}(X) \times L^2_{-\delta}(X) \to \mathbb{R}$$

defined as

$$< f, g >_\eta := \int_X f\, g\, d\,\mu_\eta$$

is $\mathcal{L}_{-\delta}$

$$\mathcal{L}_{-\delta} := \mathbb{L}_\eta : W^{4,2}_{4-\delta}(X) \to L^2_{-\delta}(X)\,.$$

Moreover $\mathcal{L}_\delta$ is a Fredholm operator and

$$\mathbb{L}_\eta : W^{4,2}_{\delta}(X) \to im\left(\mathbb{L}_\eta\right)$$

has a bounded inverse.

The proof of the above result is an easy consequence of the theory developed in Chapter 12 in [Pac08]. We focus now on asymptotic expansions of various operators on ALE spaces.

Lemma 2.3. *Let (X,η) be an ALE-Kähler space that is Ricci-flat. Then on the coordinate chart at infinity we have the following expansions*

- *for the inverse of the metric $\eta^{i\bar{j}}$*

$$\eta^{i\bar{j}} = 2\left[\delta_{i\bar{j}} - \frac{2c_\Gamma\,(m-1)}{|x|^{2m}}\left(\delta_{i\bar{j}} - m\frac{\overline{x^i}x^j}{|x|^2}\right) + \mathcal{O}\left(|x|^{-2-2m}\right)\right]\,; \tag{2.5}$$

- *the unit normal vector to the sphere $|x| = \rho$*

$$\nu = \frac{\left(x^i\partial_i + \overline{x^i}\overline{\partial}_i\right)}{|x|}\left[1 + \frac{c_\Gamma\,(m-1)^2}{|x|^{2m}}\right] + \mathcal{O}\left(|x|^{-2-2m}\right)\,; \tag{2.6}$$

- *the laplacian Δ_η*

$$\Delta_\eta = \Delta - \frac{2c_\Gamma\,(m-1)}{|x|^{2m}}\Delta + \frac{8c_\Gamma\,(m-1)\,m}{|x|^{2m+2}}\overline{x^i}x^j\partial_j\overline{\partial}_i + \mathcal{O}\left(|x|^{-2-2m}\right)\,. \tag{2.7}$$

Proof. The proof of the lemma is a series of computations. We start with the inverse of the metric η

$$\begin{aligned}
\delta_i^k &= \eta_{i\bar{j}}\eta^{k\bar{j}}\\
&= \left[\frac{\delta_{i\bar{j}}}{2} - c_\Gamma\partial_i\overline{\partial}_j|x|^{2-2m} + \mathcal{O}\left(|x|^{-2-2m}\right)\right]\eta^{k\bar{j}}\\
&= \left[\frac{\delta_{i\bar{j}}}{2} - c_\Gamma\partial_i\overline{\partial}_j|x|^{2-2m} + \mathcal{O}\left(|x|^{-2-2m}\right)\right]\left[2\delta^{k\bar{j}} + X^{k\bar{j}}\right]\\
&= \delta_i^k - 2c_\Gamma\partial_i\overline{\partial}_k|x|^{2-2m} + \frac{X^{k\bar{i}}}{2} + \mathcal{O}\left(|x|^{-2-2m}\right)\,.
\end{aligned}$$

We then have

$$\begin{aligned}
X^{k\bar{i}} &= 4c_\Gamma\partial_i\overline{\partial}_k|x|^{2-2m} + \mathcal{O}\left(|x|^{-2-2m}\right)\,,\\
\eta^{i\bar{j}} &= 2\delta^{i\bar{j}} + 4c_\Gamma\partial_j\overline{\partial}_i|x|^{2-2m} + \mathcal{O}\left(|x|^{-2-2m}\right)\,.
\end{aligned}$$

Now we expand the unit normal to the sphere $|x| = \rho$. By definition we have

$$\nu = \frac{\nabla_\eta\,|x|^2}{\left|\nabla_\eta\,|x|^2\right|_\eta}$$

and we compute the various quantities involved. We have for $f \in C^1(X,\mathbb{R})$

$$\begin{aligned}
\nabla_\eta f &= \partial^\sharp f + \overline{\partial}^\sharp f\,,\\
\partial^\sharp f &= \eta^{i\bar{j}}\overline{\partial}_j f\partial_i\,,
\end{aligned}$$

$$\begin{aligned}
\partial^\sharp |x|^2 &= \eta^{i\bar{j}} \overline{\partial}_j |x|^2 \partial_i \\
&= \eta^{i\bar{j}} x^j \partial_i \\
&= 2x^i \partial_i + 4c_\Gamma x^j \partial_j \overline{\partial}_i |x|^{2-2m} \partial_i + \mathcal{O}\left(|x|^{-1-2m}\right) \\
&= 2x^i \partial_i + 4c_\Gamma x^j \left[\frac{(1-m)}{|x|^{2m}} \delta_{j\bar{i}} + \frac{m(m-1)}{|x|^{2m+2}} \overline{x^j} x^i\right] \partial_i + \mathcal{O}\left(|x|^{-1-2m}\right) \\
&= 2x^i \partial_i + \frac{4c_\Gamma (1-m)}{|x|^{2m}} x^i \partial_i + \frac{4c_\Gamma m(m-1)}{|x|^{2m+2}} x^i \partial_i + \mathcal{O}\left(|x|^{-1-2m}\right) \\
&= 2\left[1 + \frac{2c_\Gamma (m-1)^2}{|x|^{2m}}\right] x^i \partial_i + \mathcal{O}\left(|x|^{-1-2m}\right) ,
\end{aligned}$$

$$\nabla_\eta |x|^2 = 2\left[1 + \frac{2c_\Gamma (m-1)^2}{|x|^{2m}}\right] \left(x^i \partial_i + \overline{x^i \partial_i}\right) + \mathcal{O}\left(|x|^{-1-2m}\right) ,$$

$$\begin{aligned}
\left|\nabla_\eta |x|^2\right|_\eta^2 &= 2\left|\partial^\sharp |x|^2\right|_\eta^2 \\
&= 2\eta^{i\bar{j}} \overline{\partial}_j |x|^2 \eta_{i\bar{k}} \eta^{l\bar{k}} \partial_l |x|^2 \\
&= 2\overline{\partial}_j |x|^2 \eta^{l\bar{j}} \partial_l |x|^2 \\
&= 2x^j \eta^{l\bar{j}} \overline{x^l} \\
&= 2\left[2|x|^2 + 4c_\Gamma x^j \overline{x^l} \partial_j \overline{\partial}_l |x|^{2-2m} + \mathcal{O}\left(|x|^{-2m}\right)\right] \\
&= 2\left[2|x|^2 + \frac{4c_\Gamma (1-m)}{|x|^{2m-2}} + \frac{4c_\Gamma (m-1) m}{|x|^{2m-2}} + \mathcal{O}\left(|x|^{-2m}\right)\right] \\
&= 2\left[2|x|^2 + \frac{4c_\Gamma (m-1)^2}{|x|^{2m-2}} + \mathcal{O}\left(|x|^{-2m}\right)\right] \\
&= 4|x|^2 \left[1 + \frac{2c_\Gamma (m-1)^2}{|x|^{2m}} + \mathcal{O}\left(|x|^{-2m}\right)\right] .
\end{aligned}$$

So we have

$$
\begin{aligned}
\nu &= \frac{\nabla_\eta |x|^2}{\sqrt{|\nabla_\eta |x|^2|^2}} \\
&= \frac{2\left[1+\frac{2c_\Gamma(m-1)^2}{|x|^{2m}}\right]\left(x^i\partial_i + \overline{x^i}\overline{\partial}_i\right) + \mathcal{O}\left(|x|^{-1-2m}\right)}{\sqrt{4|x|^2\left[1+\frac{2c_\Gamma(m-1)^2}{|x|^{2m}} + \mathcal{O}\left(|x|^{-2-2m}\right)\right]}} \\
&= \frac{2\left[1+\frac{2c_\Gamma(m-1)^2}{|x|^{2m}}\right]\left(x^i\partial_i + \overline{x^i}\overline{\partial}_i\right) + \mathcal{O}\left(|x|^{-1-2m}\right)}{2|x|\sqrt{\left[1+\frac{2c_\Gamma(m-1)^2}{|x|^{2m}} + \mathcal{O}\left(|x|^{-2-2m}\right)\right]}} \\
&= \frac{\left[1+\frac{2c_\Gamma(m-1)^2}{|x|^{2m}}\right]\left(x^i\partial_i + \overline{x^i}\overline{\partial}_i\right) + \mathcal{O}\left(|x|^{-1-2m}\right)}{|x|\sqrt{\left[1+\frac{2c_\Gamma(m-1)^2}{|x|^{2m}} + \mathcal{O}\left(|x|^{-2-2m}\right)\right]}} \\
&= \frac{\left(x^i\partial_i + \overline{x^i}\overline{\partial}_i\right)}{|x|}\left[\frac{\left[1+\frac{2c_\Gamma(m-1)^2}{|x|^{2m}}\right]}{\sqrt{\left[1+\frac{2c_\Gamma(m-1)^2}{|x|^{2m}} + \mathcal{O}\left(|x|^{-2-2m}\right)\right]}}\right] + \frac{\mathcal{O}\left(|x|^{-1-2m}\right)}{|x|\sqrt{\left[1+\frac{2c_\Gamma(m-1)^2}{|x|^{2m}} + \mathcal{O}\left(|x|^{-2-2m}\right)\right]}} \\
&= \frac{\left(x^i\partial_i + \overline{x^i}\overline{\partial}_i\right)}{|x|}\left[1+\frac{c_\Gamma(m-1)^2}{|x|^{2m}}\right] + \mathcal{O}\left(|x|^{-1-2m}\right) \\
&= \left[1+\frac{c_\Gamma(m-1)^2}{\rho^{2m}}\right]\partial_\rho + \mathcal{O}\left(|x|^{-2-2m}\right) .
\end{aligned}
$$

We now calculate the expansion of Δ_η

$$
\begin{aligned}
\Delta_\eta &= 2\eta^{i\bar{j}}\partial_j\overline{\partial}_i \\
&= \Delta + 8c_\Gamma \partial_j\overline{\partial_i}|x|^{2-2m}\partial_i\overline{\partial_j} + \mathcal{O}\left(|x|^{-2-2m}\right) .
\end{aligned}
$$

□

We conclude this section with an observation regarding $\ker(\mathcal{L}_\delta)$ that will be useful in Chapter 3.

Proposition 2.7. *Let (X,η) be a Ricci-flat ALE Kähler manifold that is a crepant resolution of $\mathbb{C}^m/\Gamma$ with $\Gamma \triangleleft SU(m)$, $\delta \in (m+1, m+2)$, then*

$$\ker(\mathcal{L}_\delta) = \mathbb{R} .$$

Proof. Let $f \in \ker(\mathcal{L}_\delta)$, by standard elliptic regularity we have that $f \in C^\omega_{loc}(X)$. On $X \setminus X_R \simeq (\mathbb{C}^m \setminus B_R)/\Gamma$ we consider the Fourier expansion of f

$$f = \sum_{k=0}^{+\infty} f^{(k)}(|x|)\,\phi_k$$

with $f^{(k)} \in C^{n,\alpha}_{\delta-m}\left([R,+\infty)\right)$ for any $n \in \mathbb{N}$ and this sum is $C^{n,\alpha}$-convergent on compact sets. Then, using expansions(2.5), (2.6),(2.7), we have on $X \setminus X_R$

$$\begin{aligned}
0 =&\Delta^2_\eta f\\
=&\Delta^2 f + |x|^{-2m} L_4\left(f\right) + |x|^{-1-2m} L_3\left(f\right) + |x|^{-2-2m} L_2\left(f\right)\\
=&\Delta^2 \left[\sum_{k=0}^{+\infty} f^{(k)}\left(|x|\right)\phi_k\right] + |x|^{-2m} L_4\left(f\right) + |x|^{-1-2m} L_3\left(f\right) + |x|^{-2-2m} L_2\left(f\right)\\
=&\sum_{k=0}^{+\infty} \Delta^2 \left(f^{(k)}\left(|x|\right)\phi_k\right) + |x|^{-2m} L_4\left(f\right) + |x|^{-1-2m} L_3\left(f\right) + |x|^{-2-2m} L_2\left(f\right)\\
=&\sum_{k=0}^{+\infty} \Lambda_k^2 \left(f^{(k)}\left(|x|\right)\right)\phi_k + |x|^{-2m} L_4\left(f\right) + |x|^{-1-2m} L_3\left(f\right) + |x|^{-2-2m} L_2\left(f\right)
\end{aligned}$$

with

$$\Lambda_k \varphi = \partial^2_\rho \varphi + \frac{(2m-1)}{\rho}\partial_\rho \varphi - \frac{k\left(k+2m-2\right)}{\rho^2}\varphi$$

and L_k differential operators of order k and uniformly bounded coefficients. We have the equation

$$\sum_{k=0}^{+\infty} \Lambda_k^2 \left(f^{(k)}\left(|x|\right)\right)\phi_k = -|x|^{-2m} L_4\left(f\right) - |x|^{-1-2m} L_3\left(f\right) - |x|^{-2-2m} L_2\left(f\right)$$

that implies

$$\Lambda_k^2 f^{(k)} \in C^{n,\alpha}_{\delta-3m-4}\left(\mathbb{R}\right) \qquad k \geq 0\,.$$

Suppose

$$\limsup_{|x|\to+\infty} |f| > 0\,,$$

since $f^{(k)} \in C^{n,\alpha}_{\delta-m}\left(\mathbb{R}\right)$ the only possibilities are

$$f^{(0)}\left(\rho\right) = c_0 + \varphi_0\left(\rho\right)$$

$$f^{(1)}\left(\rho\right) = \left(\rho + \varphi_1\left(\rho\right)\right)\phi_1$$

with $\varphi_0, \varphi_1 \in C^{n,\alpha}_{\delta-3m}\left(\mathbb{R}\right)$ and $c_0 \in \mathbb{R}$. But there aren't ϕ_1 that are Γ-invariant, so the only possibility is that

$$f_0\left(\rho\right) = c_0 + \varphi_0\left(\rho\right)\,.$$

We now show that f is actually constant, indeed $f - c_0 \in C^{n,\alpha}_{\delta-3m}\left(X\right)$ and

$$\mathbb{L}_\eta\left(f - c_0\right) = \frac{1}{2}\Delta^2_\eta\left(f - c_0\right) = 0$$

so by Proposition 2.6 we can conclude

$$f - c_0 \equiv 0\,.$$

□

Chapter 3

Construction of the families of metrics

In this chapter we prove the existence of the families of cscK metrics on truncated manifolds. We follow steps 5 to 15 we illustrated in Chapter 1.

3.1 Construction of the family of CscK metrics on the orbifold

In this section we work out the details of steps 5 to 12. Indeed we build the function $\mathbb{F}^{o}_{\mathbf{b},\mathbf{hk}}$. We recall that we want $\mathbb{F}^{o}_{\mathbf{b},\mathbf{hk}}$ to have the following features:

1. near points $\mathbf{p}$ should have an expansion of type

$$-\hat{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \mathcal{O}\left(\varepsilon^{2m}|z|^{4-2m}\right)$$

2. should depend on and prescribe its behavior at $\partial M_{r_\varepsilon}$;

3. the resulting metric $g_{\mathbf{b},\mathbf{hk}}$ should have constant scalar curvature on M_{r_ε}.

To fulfill these requirements we build the function $\mathbb{F}^{o}_{\mathbf{b},\mathbf{hk}}$ by blocks:

$$\mathbb{F}^{o}_{\mathbf{b},\mathbf{hk}} = \mathbb{H}^{b}_{\mathbf{hk}} + \tilde{H}^{o}_{\mathbf{hk}} + f^{o}_{\mathbf{b},\mathbf{hk}}$$

and in particular

1. $\mathbb{H}^{b}_{\mathbf{hk}}$ will be the 'skeleton' of $\mathbb{F}^{o}_{\mathbf{b},\mathbf{hk}}$ and will satisfy the first requirement;

2. $\tilde{H}^{o}_{\mathbf{hk}}$ will prescribe the behavior at $\partial M_{r_\varepsilon}$;

3. $f^{o}_{\mathbf{b},\mathbf{hk}}$ will be the correction term assuring that the scalar curvature of the new metric is constant.

We recall that we want

- $g_{\mathbf{b},\mathbf{hk}}$ to have, on $B_{2r_\varepsilon}(p_j)\setminus B_{r_\varepsilon}(p_j)$, the expansion in rescaled coordinates

$$\begin{aligned}\omega_{g_{\mathbf{b},\mathbf{hk}}} =& i\partial\overline{\partial}\left(\frac{r_\varepsilon^2|w|^2}{2}+\psi_g\left(r_\varepsilon w\right)\right)\\ &+\hat{b}_j^{2m}c_{\Gamma_j}\varepsilon^{2m}i\partial\overline{\partial}\left(-r_\varepsilon^{2-2m}|w|^{2-2m}+\frac{(m-1)\,s_g r_\varepsilon^{4-2m}}{2\,(m-2)\,m\,(m+1)}|w|^{4-2m}\right)\\ &+i\partial\overline{\partial}\left(\varepsilon^2\tilde{b}_j^2\chi_{r_0,p_j}\psi_{\eta_j}\left(\frac{r_\varepsilon w}{\tilde{b}_j\varepsilon}\right)+H^o_{h_jk_j}(w)\right)+\mathcal{O}\left(\varepsilon^{2m}r_\varepsilon^{4-2m}\right);\end{aligned}$$

- $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ to have, on $X_{\frac{R_\varepsilon}{\hat{b}_j},j}\setminus X_{\frac{R_\varepsilon}{2\hat{b}_j},j}$, the expansion in rescaled coordinates

$$\begin{aligned}\omega_{\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}} =& i\partial\overline{\partial}\left(\frac{r_\varepsilon^2|w|^2}{2}+\psi_g\left(r_\varepsilon w\right)\right)\\ &+\hat{b}_j^{2m}c_{\Gamma_j}\varepsilon^{2m}i\partial\overline{\partial}\left(-r_\varepsilon^{2-2m}|w|^{2-2m}+\frac{(m-1)\,s_g r_\varepsilon^{4-2m}}{2\,(m-2)\,m\,(m+1)}|w|^{4-2m}\right)\\ &+\varepsilon^2 i\partial\overline{\partial}\left(\hat{b}_j^2\psi_{\eta_j}\left(\frac{R_\varepsilon w}{\hat{b}_j}\right)+H^I_{\tilde{h}_j\tilde{k}_j}(w)\right)+\mathcal{O}\left(r_\varepsilon^\sigma R_\varepsilon^{-2m+\delta}\right).\end{aligned}$$

So we match perfectly (by construction) the first two lines (the terms written in blue) of the above expansions. Moreover we want that $H^o_{h_jk_j},\varepsilon^2H^I_{\tilde{h}_j\tilde{k}_j}$ as functions of ε dominate all the other terms we don't match perfectly by construction. We recall that in section 1.6 there is the guide line for the proof of Theorem 1.7 and in section 1.7 there are notations and the definitions of cutoff functions we use.

3.1.1 Construction of $\tilde{H}^o_{\mathbf{hk}}$

This is step 7. To construct the block that prescribes the behavior at $\partial M_{r_\varepsilon}$ we will use outer biharmonic extension of Γ-invariant functions on the sphere. The choice of biharmonic extensions is suggested by the shape of operator $\mathbb{L}_g$ in Kähler coordinates at a point. Its most relevant part is indeed the euclidean biharmonic operator.

We set

$$\mathcal{B}_\alpha:=\left\{(\mathbf{h},\mathbf{k})\in C^{4,\alpha}(\partial B_1)^N\times C^{2,\alpha}(\partial B_1)^N\,|\,h_j,k_j\text{ are }\Gamma_j-\text{invariant}\right\}$$

and $\mathfrak{B}(\kappa,\beta,\sigma)\subset\mathcal{B}_\alpha$ the set of functions

$$(\mathbf{h},\mathbf{k})\in\mathcal{B}_\alpha$$

such that their means $h_j^{(0)},k_j^{(0)}$

$$h_j^{(0)}:=\frac{1}{\mu\left(S^{2m-1}\right)}\int_{S^{2m-1}}h_jd\mu_0\qquad k_j^{(0)}:=\frac{1}{\mu\left(S^{2m-1}\right)}\int_{S^{2m-1}}k_jd\mu_0$$

satisfy the estimate

$$\left|h_j^{(0)}\right|,\left|k_j^{(0)}\right|\leq\kappa r_\varepsilon^\beta$$

and their "non-radial parts" $h_j^{(\dagger)},k_j^{(\dagger)}$

$$h_j^{(\dagger)}:=h_j-h_j^{(0)}\qquad k_j^{(\dagger)}:=k_j-k_j^{(0)}$$

satisfy the estimate
$$\left\|h_j^{(\dagger)}\right\|_{C^{4,\alpha}(S^{2m-1})}, \left\|k_j^{(\dagger)}\right\|_{C^{2,\alpha}(S^{2m-1})} \leq \kappa r_\varepsilon^\sigma$$
with
$$r_\varepsilon^\sigma = \varepsilon^{2m+4} r_\varepsilon^{-2-2m-\tau} \qquad \tau > 0$$
$$r_\varepsilon^\beta = \varepsilon^{4m+2} r_\varepsilon^{-4m-\tau} \qquad \tau > 0$$
and $\kappa \in \mathbb{R}^+$ to be determined. From now on $(\mathbf{h}, \mathbf{k}) \in \mathfrak{B}(\kappa, \beta, \sigma)$. We indicate H_{hk}^o the biharmonic extension on $\mathbb{C}^m \setminus B_1$ of functions $h, k \in \mathcal{B}_\alpha$ gven by the solution of the boundary problem
$$\begin{cases} \Delta^2 H_{hk}^o = 0 & w \in \mathbb{C}^m \setminus B_1 \\ H_{hk}^o = h & w \in \partial B_1 \\ \Delta H_{hk}^o = k & w \in \partial B_1 \end{cases}$$
and it has the following expansion
$$H_{h,k}^o := \sum_{\gamma=0}^{+\infty} \left(\left(h^{(\gamma)} + \frac{k^{(\gamma)}}{4(m+\gamma-2)} \right) |w|^{2-2m-\gamma} - \frac{k^{(\gamma)}}{4(m+\gamma-2)} |w|^{4-2m-\gamma} \right) \phi_\gamma .$$
We recall that with $h^{(\gamma)}\phi_\gamma, k^{(\gamma)}\phi_\gamma$ we mean the projection of h, k onto the γ-th eigenspace of $\Delta_{S^{2m-1}}$ with the orthonormal basis $\left\{\bar{\phi}_{\gamma,1}, \ldots, \bar{\phi}_{\gamma,N_\gamma}\right\}$. We recall also that if the group Γ is non trivial then we have no ϕ_1 in the above summations. We define
$$\hat{H}_{hk}^o := \sum_{\gamma=2}^{+\infty} \left(\left(h^{(\gamma)} + \frac{k^{(\gamma)}}{4(m+\gamma-2)} \right) |w|^{2-2m-\gamma} - \frac{k^{(\gamma)}}{4(m+\gamma-2)} |w|^{4-2m-\gamma} \right) \phi_\gamma$$
and we can finally construct $\tilde{H}_{\mathbf{hk}}^o \in C^{4,\alpha}(M_{r_\varepsilon})$
$$\tilde{H}_{\mathbf{hk}}^o := \sum_{j=1}^{N} \chi_{j,r_0} \hat{H}_{h_j,k_j}^o \left(\frac{z}{r_\varepsilon} \right) .$$
We didn't involve
$$\mathbf{h}^{(0)} := \left(h_1^{(0)}, \ldots, h_N^{(0)} \right)$$
$$\mathbf{k}^{(0)} := \left(k_1^{(0)}, \ldots, k_N^{(0)} \right)$$
in the construction of $\tilde{H}_{\mathbf{hk}}^o$ on purpose, this is a trick to have better estimates when we will look for $f_{\mathbf{b},\mathbf{hk}}^o$. We will use $\mathbf{h}^{(0)}, \mathbf{k}^{(0)}$ in the construction of the skeleton solution

3.1.2 Construction of the skeleton $\mathbb{H}_{\mathbf{hk}}^b$

Let (M, g) be the base m-dimensional cscK orbifold with isolated singularities and let
$$\mathbf{p} = \{p_1, \ldots, p_N\}$$
be its N singular points admitting a crepant resolution. We fix once and for all a L^2-orthonormal basis for $\ker(\mathbb{L}_g)$
$$\ker(\mathbb{L}_g) = \langle 1, \varphi_1, \ldots, \varphi_d \rangle .$$
Suppose points $\mathbf{p}$ satisfy the following conditions:

1. $N \geq d+1 = \dim\left(\ker\left(\mathbb{L}_g\right)\right)$;
2. the $d \times N$ matrix
$$\Delta\Phi\left(\mathbf{p}\right)_{i,j} := \left[\Delta_g \varphi_i\left(p_j\right) + s_g \varphi_i\left(p_j\right)\right]$$
has full rank and so there exist right inverses for $\Delta\Phi\left(\mathbf{p}\right)$. We now choose a particular one
$$\Delta\Phi\left(\mathbf{p}\right)_r^{-1} := \Delta\Phi\left(\mathbf{p}\right)^* \left(\Delta\Phi\left(\mathbf{p}\right)\Delta\Phi\left(\mathbf{p}\right)^*\right)^{-1} ;$$
3. there exist $\mathbf{b} := \left(b_1, \ldots, b_N\right) \in \mathbb{R}^N_+$ s.t.
$$\Delta\Phi\left(\mathbf{p}\right)\mathbf{b} = 0\,.$$

From now on we will call these set of conditions **BAL(p)** with coefficients **b**.

Remark 3.1. We want to point out that the conditions above are a special case of conditions 2.3 and 2.4 of Proposition 2.5 and will allow us to construct an inverse for the operator $\mathbb{L}_g$ between suitable spaces.

We are ready for step 6 that is the following lemma.

Lemma 3.1. *Suppose we have $\boldsymbol{p}$ that satisfy **BAL(p)** with coefficients $\boldsymbol{b}$.For $\varepsilon << 1$ we can find a function $H^{\boldsymbol{b}} \in C^\infty_{loc}\left(M_{\boldsymbol{p}}\right)$ with the following properties:*

- *satisfies in $M_{\boldsymbol{p}}$*
$$\mathbb{L}_g H^{\boldsymbol{b}} - \frac{(m-1)\, s_g}{Vol_g\left(M\right)} \sum_{j=1}^{N} b_j c_{\Gamma_j} \frac{\mu\left(S^{2m-1}\right)}{\Gamma_j} = 0\,;$$
- *near points of $\boldsymbol{p}$ has expansion*
$$\begin{aligned} H^{\boldsymbol{b}}\left(z\right) = &- b_j c_{\Gamma_j}\left(|z|^{2-2m} - \frac{(m-1)\, s_g}{2m\left(m+1\right)\left(m-2\right)}|z|^{4-2m}\right) \\ &+ \left(\phi_2 + \phi_4\right)|z|^{4-2m} + |z|^{5-2m}\sum_{h=1}^{K}\tilde{\phi}_{2h+1} + \mathcal{O}\left(|z|^{6-2m}\right) \end{aligned}$$

with ϕ_q eigenfunctions of $\Delta_{S^{2m-1}}$ relative to eigenvalues $-q\left(2m-2+q\right)$.

Proof. Finding a function $H^{\mathbf{b}}$ with the above properties means solving the equation
$$\mathbb{L}_g u - \frac{(m-1)\, s_g}{\mathrm{Vol}_g\left(M\right)} \sum_{j=1}^{N} b_j c_{\Gamma_j} \frac{\mu\left(S^{2m-1}\right)}{|\Gamma_j|} = -\left(m-1\right)\sum_{j=1}^{N} b_j c_{\Gamma_j}\frac{\mu\left(S^{2m-1}\right)}{|\Gamma_j|}\left[\Delta_g \delta_{p_j} + s_g \delta_{p_j}\right]\,.$$

This equation has solution if its right hand side is "L^2-orthogonal" to $\ker\left(\mathbb{L}_g\right)$. This translates to the linear system of equations
$$\Delta\Phi\left(\mathbf{p}\right)\mathbf{b} = \mathbf{0}$$
that are satisfied by assumptions, so by Proposition 2.5 we have the existence of $H^{\mathbf{b}}$. □

From now on we will denote
$$S_g\left(\mathbf{b}\right) := \frac{(m-1)\, s_g}{\mathrm{Vol}_g\left(M\right)} \sum_{j=1}^{N} b_j c_{\Gamma_j} \frac{\mu\left(S^{2m-1}\right)}{|\Gamma_j|}\,.$$

As we explained in step 7, we modify a little $H^{\mathbf{b}}$.

Lemma 3.2. *Suppose we have* $\boldsymbol{p}$ *that satisfy* $\boldsymbol{BAL(p)}$ *with coefficients* $\boldsymbol{b}$*. Let* $\boldsymbol{h}^{(0)}, \boldsymbol{k}^{(0)} \in \mathbb{R}^N$ *such that*

$$\left|h_j^{(0)}\right|, \left|k_j^{(0)}\right| \leq \kappa r_\varepsilon^\beta .$$

We can find

$$\tilde{\boldsymbol{b}} \in \mathbb{R}^N$$

and a function $H_{\boldsymbol{hk}}^b \in C_{loc}^\infty(M_{\boldsymbol{p}}) \cap C_{2-2m}^{4,\alpha}(M_{\boldsymbol{p}})$ *with the following properties.*

- *Coefficients* $\tilde{\boldsymbol{b}}$ *satisfy* $|\tilde{\boldsymbol{b}} - \boldsymbol{b}| < C r_\varepsilon^{2m-4+\beta} \varepsilon^{-2m}$.
- *The function* $H_{\boldsymbol{hk}}^b$ *satisfies in* $M_{\boldsymbol{p}}$

$$\mathbb{L}_g H_{\boldsymbol{hk}}^b - S_g(\boldsymbol{b}) + S_g^{'}\left(\boldsymbol{b}, \boldsymbol{h}^{(0)}, \boldsymbol{k}^{(0)}\right) = 0$$

with $\left|S_g^{'}\left(\boldsymbol{b}, \boldsymbol{h}^{(0)}, \boldsymbol{k}^{(0)}\right)\right| < C r_\varepsilon^{2m-4+\beta} \varepsilon^{-2m}$.

- *The function* $H_{\boldsymbol{hk}}^b$ *satisfies near points of* $\boldsymbol{p}$ *has expansion*

$$\begin{aligned} H_{\boldsymbol{hk}}^b(z) = & - c_{\Gamma_j} \tilde{b}_j \left(|z|^{2-2m} - \frac{(m-1)\, s_g}{2m(m+1)(m-2)} |z|^{4-2m} \right) \\ & + \left(h_j^{(0)} + \frac{k_j^{(0)}}{4(m-2)} \right) r_\varepsilon^{2m-2} \varepsilon^{-2m} |z|^{2-2m} \\ & - \frac{k_j^{(0)}}{4(m-2)} r_\varepsilon^{2m-4} \varepsilon^{-2m} |z|^{4-2m} \\ & + (\phi_2 + \phi_4) |z|^{4-2m} + |z|^{5-2m} \sum_{h=1}^{K} \phi_{2h+1} + \mathcal{O}\left(|z|^{6-2m}\right) \end{aligned}$$

with ϕ_q *eigenfunctions of* $\Delta_{S^{2m-1}}$ *relative to eigenvalues* $-q(2m-2+q)$.

Proof. We set $\tilde{\mathbf{b}} = \mathbf{b} + \mathbf{b}'$ and

$$C(\varepsilon) := \frac{(m-1)\, r_\varepsilon^{2m-4} \mu\left(S^{2m-1}\right)}{\varepsilon^{2m} \mathrm{Vol}_g(M)},$$

$$S_g^{'}\left(\mathbf{b}, \mathbf{h}^{(0)}, \mathbf{k}^{(0)}\right) := - S_g(\mathbf{b}') + \sum_{j=1}^{N} \frac{C(\varepsilon)}{|\Gamma_j|} \left[\frac{(m^2 - m + 2)\, s_g r_\varepsilon^2}{m(m+1)} \left(h_j^{(0)} + \frac{k_j^{(0)}}{4(m-2)} \right) + k_j^{(0)} \right].$$

We look for a function such that

$$\begin{aligned} \mathbb{L}_g u = & - S_g^{'}\left(\mathbf{b}, \mathbf{h}^{(0)}, \mathbf{k}^{(0)}\right) - (m-1)\, \mu\left(S^{2m-1}\right) \sum_{j=1}^{N} \frac{c_{\Gamma_j} \tilde{b}_j}{|\Gamma_j|} \left(\Delta_g \delta_{p_j} + s_g \delta_{p_j}\right) \\ & + C(\varepsilon)\, r_\varepsilon^2 \sum_{j=1}^{N} \frac{1}{|\Gamma_j|} \left(h_j^{(0)} + \frac{k_j^{(0)}}{4(m-2)} \right) \left(\Delta_g \delta_{p_j} + \frac{(m^2 - m + 2)\, s_g}{m(m+1)} \delta_{p_j} \right) \\ & + C(\varepsilon) \sum_{j=1}^{N} \frac{k_j^{(0)}}{|\Gamma_j|} \delta_{p_j} . \end{aligned}$$

We define

$$\mathbf{c} := \begin{pmatrix} b_j' \\ \Gamma_j \end{pmatrix}$$

and to solve the above PDE we only have to find a solution to a linear system of the form

$$\Delta\Phi\mathbf{c} + r_\varepsilon^{2m-2}\varepsilon^{-2m}M_1\left(\mathbf{h}^{(0)} + \frac{\mathbf{k}^{(0)}}{4(m-2)}\right) + r_\varepsilon^{2m-4}\varepsilon^{-2m}M_2\mathbf{k}^{(0)} = \mathbf{0}$$

with M_1, M_2 real $d \times N$ matrices not depending on ε. Using the right inverse $\Delta\Phi\left(\mathbf{p}\right)_r^{-1}$ we have

$$\mathbf{c} = -r_\varepsilon^{2m-2}\varepsilon^{-2m}\Delta\Phi\left(\mathbf{p}\right)_r^{-1}M_1\left(\mathbf{h}^{(0)} + \frac{\mathbf{k}^{(0)}}{4(m-2)}\right) + r_\varepsilon^{2m-4}\varepsilon^{-2m}\Delta\Phi\left(\mathbf{p}\right)_r^{-1}M_2\mathbf{k}^{(0)}$$

and

$$|\mathbf{b}'| < r_\varepsilon^{2m-4+\beta}\varepsilon^{-2m}.$$

By Proposition 2.5 we have the existence of u and setting

$$H^b_{\mathbf{hk}} := H^b + u$$

we get our desired function. □

Remark 3.2. We want to point out that the quantities

$$\left(h_j^{(0)} + \frac{k_j^{(0)}}{4(m-2)}\right)r_\varepsilon^{2m-2}\varepsilon^{-2m}, \quad \frac{k_j^{(0)}}{4(m-2)}r_\varepsilon^{2m-4}\varepsilon^{-2m}$$

are positive powers of ε and so we have

$$\lim_{\varepsilon\to+0}\tilde{b}_j = b_j.$$

Let now (X_j, η_j) the Ricci flat ALE Kähler space associated to the p_j. On $X_j \setminus X_{R_0,j}$ we have a potential for the metric η_j of the form

$$\frac{|x|^2}{2} - c_{\Gamma_j}|x|^{2-2m} + \psi_{\eta_j}(x),$$
$$\psi_{\eta_j}(x) = \mathcal{O}\left(|x|^{-2m}\right).$$

As explained in step 8 we build our skeleton solution, we set

$$\bar{b}_j := \sqrt[2m]{\tilde{b}_j},$$
$$\mathbb{H}^b_{\mathbf{hk}} := \varepsilon^{2m}H^b_{\mathbf{hk}} + \sum_{j=1}^{N}\varepsilon^2\bar{b}_j^2\tilde{\chi}_{j,r_0}\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right) \tag{3.1}$$

This is a "natural choice" of perturbation, since in a small ball $B_{r_0}(p_j)$ the operator $\mathbf{s}_g(\cdot)$ is very close (in some suitable sense) to $\mathbf{s}_0(\cdot)$, and we have by assumptions that

$$\mathbf{s}_0\left(-c_{\Gamma_j}|z|^{2-2m} + \psi_{\eta_j}\right) = 0 \quad \text{on } \mathbb{C}^m \setminus B_{R_0}.$$

So, in the small annuli $B_{r_0} \setminus B_{r_\varepsilon}$, we correct our background metric with something that doesn't affect so much the scalar curvature, indeed it stays very close to the initial one. This construction has been inspired to the one Szekelyhidi performs in [Szé12], in the case of blow ups.

3.1.3 Construction of $f^o_{\mathbf{b},\mathbf{hk}}$ and the fixed point argument

We have come to step 10. Now that we have $\mathbb{H}^b_{\mathbf{hk}}$ and $\tilde{H}^o_{\mathbf{hk}}$ we want to find the last term. To do this we now derive an equation for $f^o_{\mathbf{b},\mathbf{hk}}$ and we will solve it by using the Banach-Caccioppoli fixed point theorem in suitable Banach Spaces. We start from

$$\omega_{g_{\mathbf{b},\mathbf{hk}}} = \omega_g + i\partial\overline{\partial}\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right)$$

and we want that

$$s_{g_{\mathbf{b},\mathbf{hk}}} = s_g + \nu \qquad \text{with } \nu \in \mathbb{R}$$

on M_{r_ε}. Since we want to construct $\omega_{g_{\mathbf{b},\mathbf{hk}}}$ as a small perturbation of ω_g we can expand the term in the left hand side and we have

$$s_g - \mathbb{L}_g\mathbb{H}^b_{\mathbf{hk}} - \mathbb{L}_g\tilde{H}^o_{\mathbf{hk}} - \mathbb{L}_g f^o_{\mathbf{b},\mathbf{hk}} + Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right) = s_g + \nu\,.$$

After simplifying and reordering terms we have

$$\begin{aligned}\mathbb{L}_g f^o_{\mathbf{b},\mathbf{hk}} + \nu =& - \mathbb{L}_g\tilde{H}^o_{\mathbf{hk}} - \mathbb{L}_g\mathbb{H}^b_{\mathbf{hk}} + Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right)\\ =& - \mathbb{L}_g\tilde{H}^o_{\mathbf{hk}} - \mathbb{L}_g\left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m}H^b_{\mathbf{hk}}\right) - \varepsilon^{2m}S_g\left(\mathbf{b}\right) + \varepsilon^{2m}S_g^{'}\left(\mathbf{b},\mathbf{h}^{(0)},\mathbf{k}^{(0)}\right)\\ &+ Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right)\,.\end{aligned}$$

We now set

$$\nu' := \nu + \varepsilon^{2m}S_g\left(\mathbf{b}\right) - \varepsilon^{2m}S_g^{'}\left(\mathbf{b},\mathbf{h}^{(0)},\mathbf{k}^{(0)}\right)$$

and so we want to solve on M_{r_ε} the following equation

$$\mathbb{L}_g f^o_{\mathbf{b},\mathbf{hk}} + \nu' = -\mathbb{L}_g\tilde{H}^o_{\mathbf{hk}} - \mathbb{L}_g\left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m}H^b_{\mathbf{hk}}\right) + Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right)\,.$$

We have to choose a functional space in which we can solve this equation and such that we can gain some informations on the ε-growth of the solution. We would like to to set up our problem in weighted Hölder spaces $C^{4,\alpha}_{4-2m+\tau}\left(M_{\mathbf{p}}\right)$ but we have an equation defined only on M_{r_ε}. To overcome this difficulty we introduce a truncation/extension operator between weighted spaces.

Definition 3.1. Let $f \in C^{0,\alpha}_\delta\left(M\right)$ we define $\mathcal{E}_{r_\varepsilon} : C^{0,\alpha}_\delta\left(M\right) \to C^{0,\alpha}_\delta\left(M\right)$

$$\mathcal{E}_{r_\varepsilon}\left(f\right) : \begin{cases} f\left(z\right) & z \in B_{2r_\varepsilon}\setminus B_{r_\varepsilon}\\ f\left(r_\varepsilon\frac{z}{|z|}\right)\chi_1\left(\frac{|z|}{r_\varepsilon}\right) & z \in B_{r_\varepsilon}\setminus B_{r_\varepsilon/2}\\ 0 & z \in B_{r_\varepsilon/2}\end{cases}$$

The equation we want to solve is

$$\mathbb{L}_g f^o_{\mathbf{b},\mathbf{hk}} + \nu' = -\mathcal{E}_{r_\varepsilon}\mathbb{L}_g\tilde{H}^o_{\mathbf{hk}} - \mathcal{E}_{r_\varepsilon}\mathbb{L}_g\left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m}H^b_{\mathbf{hk}}\right) + \mathcal{E}_{r_\varepsilon}Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right)\,. \tag{3.2}$$

To carry on our constructon we need the following (crucial) technical result. This is a consequence of Proposition 2.5 indeed we use a specialized form of conditions (2.3) and (2.4).

Proposition 3.1. *Let points* $\boldsymbol{p}$ *satisfy* ***BAL(p)***, *let* $\delta \in (4-2m, 0)$ *and* $f \in C^{0,\alpha}_{\delta-4}(M_{\boldsymbol{p}})$. *Then there exist* $u^{\perp} \in C^{4,\alpha}_{\delta}(M_{\boldsymbol{p}}) \cap \ker(\mathbb{L}_g)^{\perp}$ *such that*

$$\mathbb{L}_g\left(u^{\perp}\right) = f^{\perp}$$

with

$$f^{\perp} := f - \frac{1}{Vol_g(M)} \int_M f d\mu_g - \sum_{i=1}^{d} \varphi_i \int_M f \varphi_i d\mu_g$$

and $H \in C^{\infty}_{2-2m}(M_{\boldsymbol{p}}) \cap C^{\infty}_{loc}(M_{\boldsymbol{p}})$ *such that on* $M_{\boldsymbol{p}}$ *satisfies*

$$\mathbb{L}_g(H) + d_0(f) = \sum_{j=1}^{N} d_j(f) c_{\Gamma_j} \frac{\mu\left(S^{2m-1}\right)}{|\Gamma_j|}\left[\Delta_g \delta_{p_j} + s_g \delta_{p_j}\right] + \sum_{l=1}^{d} \varphi_l \int_M f \varphi_l d\mu_g$$

with $d_0(f), \ldots, d_N(f) \in \mathbb{R}$ *depending on* f. *Moreover we have the estimates*

$$\begin{aligned}\left\|u^{\perp}\right\|_{C^{4,\alpha}_{\delta}(M_{\boldsymbol{p}})} &\leq C(g,\delta)\|f\|_{C^{0,\alpha}_{\delta-4}(M_{\boldsymbol{p}})}\,, \\ \|H\|_{C^{4,\alpha}_{2-2m}(M_{\boldsymbol{p}})} &\leq C(g,\delta)\|f\|_{C^{0,\alpha}_{\delta-4}(M_{\boldsymbol{p}})}\,.\end{aligned}$$

Proof. We want to solve on $M_{\mathbf{p}}$ the equation

$$\mathbb{L}_g(u) = f - \frac{1}{\mathrm{Vol}_g(M)} \int_M f d\mu_g$$

and to reach this goal we will split this equation in a couple of equations. Let

$$f^{\perp} := f - \frac{1}{\mathrm{Vol}_g(M)} \int_M f d\mu_g - \sum_{i=1}^{d} \varphi_i \int_M f \varphi_i d\mu_g$$

then by Proposition 2.2 there exist $u^{\perp} \in C^{4,\alpha}_{\delta}(M_{\mathbf{p}}) \cap \ker(\mathbb{L}_g)^{\perp}$

$$\mathbb{L}_g\left(u^{\perp}\right) = f^{\perp}$$

and

$$\left\|u^{\perp}\right\|_{C^{4,\alpha}_{\delta}(M_{\mathbf{p}})} \leq C\|f\|_{C^{0,\alpha}_{\delta-4}(M_{\mathbf{p}})}\,.$$

The other equation we want to solve is

$$\mathbb{L}_g(H) + d_0 = \sum_{j=1}^{N} d_j c_{\Gamma_j} \frac{\mu\left(S^{2m-1}\right)}{|\Gamma_j|}\left[\Delta_g \delta_{p_j} + s_g \delta_{p_j}\right] + \sum_{l=1}^{d} \varphi_l \int_M f \varphi_l d\mu_g$$

for some constants d_j that we want to determine. Since $\mathbf{BAL(p)}$ is satisfied we can use $\Delta\Phi(\mathbf{p})_r^{-1}$ to find d_j such that the right hand side of the above equation is orthogonal to $\ker(\mathbb{L}_g)$, indeed

$$d_j = \left(\Delta\Phi(\mathbf{p})_r^{-1}\right)_{jk} \int_M f \varphi_k d\mu_g$$

do the job. Now the existence of H and its estiamtes follow from Proposition 2.5. □

In the preceding proposition we used particular functions in $C^{\infty}_{loc}(M_{\mathbf{p}})$ to make $\mathbb{L}_g$ surjective on functions on $f \in C^{0,\alpha}_{\delta-4}(M_{\mathbf{p}})$ with vanishing mean. We now select a finite dimensional subspace $\mathcal{D} \subset C^{\infty}_{2-2m}(M_{\mathbf{p}})$ that will make $\mathbb{L}_g$ surjective on functions on $f \in C^{0,\alpha}_{\delta-4}(M_{\mathbf{p}})$ with vanishing mean. Let $\mathbf{c} \in \mathbb{R}^d$ fixed, we define $\mathbf{d} \in \mathbb{R}^N$ as

$$\mathbf{d} = \Delta\Phi(\mathbf{p})_r^{-1}\,\mathbf{c}$$

and so the equation

$$\mathbb{L}_g(H_{\mathbf{c}}) + \frac{(m-1)\,s_g}{\mathrm{Vol}_g(M)}\sum_{j=1}^{N} d_j c_{\Gamma_j}\frac{\mu\left(S^{2m-1}\right)}{|\Gamma_j|} = \sum_{j=1}^{N} d_j c_{\Gamma_j}\frac{\mu\left(S^{2m-1}\right)}{|\Gamma_j|}\left[\Delta_g\delta_{p_j} + s_g\delta_{p_j}\right] + \sum_{l=1}^{d} c_l\varphi_l$$

is solvable with a unique solution $H_{\mathbf{c}}$.

Definition 3.2. We define the deficiency space $\mathcal{D} \subset C^{4,\alpha}_{2-2m}(M_{\mathbf{p}}) \cap C^{\infty}_{loc}(M_{\mathbf{p}})$

$$\mathcal{D} := \left\{H_{\mathbf{c}} \in C^{4,\alpha}_{2-2m}(M_{\mathbf{p}}) \cap C^{\infty}_{loc}(M_{\mathbf{p}}) \mid \mathbf{c} \in \mathbb{R}^d\right\}$$

so $\mathcal{D}$ is a real d-dimensional vector space. We use $\mathcal{D}$ to “extend” $C^{4,\alpha}_{\delta-4}(M_{\mathbf{p}})$, that is, we consider the space

$$C^{4,\alpha}_{\delta}(M_{\mathbf{p}}) \oplus \mathcal{D}\,.$$

We define a norm of for this space in the following manner, let $f + H_{\mathbf{c}} \in C^{0,\alpha}_{\delta}(M_{\mathbf{p}}) \oplus \mathcal{D}$ then

$$\|f + H_{\mathbf{c}}\|_{C^{0,\alpha}_{\delta}(M_{\mathbf{p}})\oplus\mathcal{D}} := \|f\|_{C^{0,\alpha}_{\delta}(M_{\mathbf{p}})} + \|H_{\mathbf{c}}\|_{C^{4,\alpha}_{2-2m}(M_{\mathbf{p}})}$$

We can rephrase Proposition 3.1 in the following form.

Proposition 3.2. *Let points* $\boldsymbol{p}$ *satisfy* ***BAL(p)****, let* $\delta \in (4-2m, 0)$ *and* $\left(C^{0,\alpha}_{\delta-4}(M_{\boldsymbol{p}})\right)_0 \subseteq C^{0,\alpha}_{\delta-4}(M_{\boldsymbol{p}})$ *the set of functions* $f \in C^{0,\alpha}_{\delta-4}(M_{\boldsymbol{p}})$ *such that*

$$\int_M f\, d\mu_g = 0\,.$$

There exist a continuous inverse for $\mathbb{L}_g$

$$\mathbb{G}_\delta : \left(C^{0,\alpha}_{\delta-4}(M_{\boldsymbol{p}})\right)_0 \to C^{4,\alpha}_{\delta}(M_{\boldsymbol{p}}) \oplus \mathcal{D}$$

with

$$\|\mathbb{G}_\delta(f)\|_{C^{4,\alpha}_{\delta}(M_{\boldsymbol{p}})\oplus\mathcal{D}} \leq C(g,\delta)\,\|f\|_{C^{0,\alpha}_{\delta-4}(M_{\boldsymbol{p}})}\,.$$

In light of proposition 3.2 we want to solve equation (3.2) for a particular value of ν', that is

$$\nu' = \frac{1}{\mathrm{Vol}_g(M)}\int_M \left[-\mathcal{E}_{r_\varepsilon}\mathbb{L}_g\tilde{H}^{o}_{\mathbf{hk}} - \mathcal{E}_{r_\varepsilon}\mathbb{L}_g\left(\mathbb{H}^{b}_{\mathbf{hk}} - \varepsilon^{2m}H^{b}_{\mathbf{hk}}\right) + \mathcal{E}_{r_\varepsilon}Q_g\left(\mathbb{H}^{b}_{\mathbf{hk}} + \tilde{H}^{o}_{\mathbf{hk}} + f^{o}_{\mathbf{b},\mathbf{hk}}\right)\right] d\mu_g$$

From now on, our weight δ we will be of the form

$$\delta = 4 - 2m + \tau \qquad \tau \in \left(0, \frac{1}{(m+2)^2}\right).$$

We make this choice of τ in order to get, when we will perform the data matching, estimates (4.5) and (4.6). To find our $f^o_{\mathbf{b},\mathbf{hk}}$ we define a nonlinear operator

$$\mathcal{N} : C^{4,\alpha}_{4+\tau-2m}\left(M_{\mathbf{p}}\right) \oplus \mathcal{D} \times \mathfrak{B}\left(\kappa,\beta,\sigma\right) \to C^{4,\alpha}_{4+\tau-2m}\left(M_{\mathbf{p}}\right) \oplus \mathcal{D}\,,$$

$$\mathcal{N}\left(f,\mathbf{h},\mathbf{k}\right) = \mathbb{G}_\tau\left[-\mathcal{E}_{r_\varepsilon}\mathbb{L}_g\tilde{H}^o_{\mathbf{hk}} - \mathcal{E}_{r_\varepsilon}\mathbb{L}_g\left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m}H^b_{\mathbf{hk}}\right) + \mathcal{E}_{r_\varepsilon}Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f\right) - \nu'\left(f\right)\right]$$

with

$$\mathbb{G}_\tau := \mathbb{G}_{4+\tau-2m} : \left(C^{0,\alpha}_{\tau-2m}\left(M_{\mathbf{p}}\right)\right)_0 \to C^{4,\alpha}_{4+\tau-2m}\left(M_{\mathbf{p}}\right) \oplus \mathcal{D}$$

the map introduced in Proposition 3.2 and

$$\nu'\left(f\right) := \frac{1}{\mathrm{Vol}_g\left(M\right)}\int_M \left[-\mathcal{E}_{r_\varepsilon}\mathbb{L}_g\tilde{H}^o_{\mathbf{hk}} - \mathcal{E}_{r_\varepsilon}\mathbb{L}_g\left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m}H^b_{\mathbf{hk}}\right) + \mathcal{E}_{r_\varepsilon}Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f\right)\right] d\mu_g\,.$$

To reach our goal we have to find a fixed point for $\mathcal{N}$. The main tool we'll use to do this is the Banach-Caccioppoli fixed point theorem. If we fix $(\mathbf{h},\mathbf{k}) \in \mathcal{B}_\alpha$ we need to find a subset (a ball) of $C^{4,\alpha}_{4+\tau-2m}\left(M_{\mathbf{p}}\right) \oplus \mathcal{D}$ in which $\mathcal{N}\left(\cdot,\mathbf{h},\mathbf{k}\right)$ is contractive. We now estimate

- $\left\|\mathcal{N}\left(0,\mathbf{h},\mathbf{k}\right)\right\|_{C^{4,\alpha}_{4+\tau-2m}\left(M_{\mathbf{p}}\right)\oplus\mathcal{D}}$
- the lipschitz constant of $\mathcal{N}\left(\cdot,\mathbf{h},\mathbf{k}\right)$ on a domanin of $C^{4,\alpha}_{4+\tau-2m}\left(M_{\mathbf{p}}\right) \oplus \mathcal{D}$
- the lipschitz constant of $\mathcal{N}\left(f,\cdot,\cdot\right)$ on $\mathfrak{B}\left(\kappa,\beta,\sigma\right)$

To get more refined estimates we will consider this equivalent expression for $\mathcal{N}$

$$\begin{aligned}\mathcal{N}\left(f,\mathbf{h},\mathbf{k}\right) = \mathbb{G}_\tau\Big[&-\mathcal{E}_{r_\varepsilon}\mathbb{L}_g\tilde{H}^o_{\mathbf{hk}} + \mathcal{E}_{r_\varepsilon}\left(-\mathbb{L}_g\left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m}H^b_{\mathbf{hk}}\right) + Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right)\right)\\ &+\mathcal{E}_{r_\varepsilon}\left(Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f\right) - Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right)\right) + \nu'\left(f\right)\Big]\end{aligned}$$

Definition 3.3. Let Ω be an open domain with compact closure of a smooth manifold and let $f \in C^{k,\alpha}\left(\Omega\right)$. We denote with $\mathcal{O}_{C^{k,\alpha}(\Omega)}\left(f\right)$ a generic $C^{k,\alpha}\left(\Omega\right)$ function such that

$$\left\|\mathcal{O}_{C^{k,\alpha}(\Omega)}\left(f\right)\right\|_{C^{k,\alpha}(\Omega)} = \left\|f\right\|_{C^{k,\alpha}(\Omega)}\,.$$

We begin with the following.

Lemma 3.3. *Let* $(\boldsymbol{h},\boldsymbol{k}) \in \mathfrak{B}\left(\kappa,\beta,\sigma\right)$*, then we have*

$$\left\|\mathcal{E}_{r_\varepsilon}\mathbb{L}_g\tilde{H}^o_{\boldsymbol{hk}}\right\|_{C^{0,\alpha}_{\tau-2m}(M_{\boldsymbol{p}})} \le C\left(g\right)\left\|\boldsymbol{h}^\dagger,\boldsymbol{k}^\dagger\right\|_{\mathcal{B}_\alpha} r_\varepsilon^{2m-(2+\tau)}\,.$$

Proof. We recall that in a ball $B_{2r_0}(p_i)$ we have that

$$\mathbb{L}_g = \frac{\Delta^2}{2} + \widetilde{\mathbb{L}}_g\,.$$

We want to estimate

$$\left\|\mathcal{E}_{r_\varepsilon}\mathbb{L}_g\tilde{H}^o_{\mathbf{hk}}\right\|_{C^{0,\alpha}_{\tau-2m}(M_{\mathbf{p}})}\,.$$

We compute

$$\mathbb{L}_g\chi_{j,r_0}\hat{H}^o_{h_jk_j} = \frac{1}{2}\Delta^2\chi_{j,r_0}\hat{H}^o_{h_jk_j} + \widetilde{\mathbb{L}}_g\chi_{j,r_0}\hat{H}^o_{h_jk_j}\,.$$

We recall that

$$\hat{H}^{o}_{h,k}(w) = \sum_{\gamma=2}^{+\infty} \left(\left(h^{\gamma} + \frac{k^{\gamma}}{4(m-2+\gamma)} \right) |w|^{2-2m-\gamma} - \frac{k^{\gamma}}{4(m-2+\gamma)} |w|^{4-2m-\gamma} \right) \phi_{\gamma} \,.$$

On M_{r_0} we have

$$\mathbb{L}_g \chi_{j,r_0} \hat{H}^{o}_{h_j k_j} = r_{\varepsilon}^{2m-2+\sigma} \mathcal{O}_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)}(1)$$

and so

$$\left\| \mathbb{L}_g \tilde{H}^{o}_{\mathbf{hk}} \right\|_{C^{0,\alpha}\left(M_{r_0}\right)} \leq C(g)\, \kappa r_{\varepsilon}^{2m-2+\sigma} \,.$$

Now we give an estimate of the weighted part of the norm. On $B_{2\rho}(p_j) \setminus B_{\rho}(p_j)$ we have

$$\begin{aligned}
\mathbb{L}_g \tilde{H}^{o}_{\mathbf{hk}}(\rho w) =& \mathbb{L}_g \hat{H}^{o}_{h_j k_j}\left(\frac{\rho w}{r_{\varepsilon}} \right) \\
=& \widetilde{\mathbb{L}}_g \hat{H}^{o}_{h_j k_j}\left(\frac{\rho w}{r_{\varepsilon}} \right) \\
=& r_{\varepsilon}^{2m-2+\sigma} \rho^{-2m} \mathcal{O}_{C^{1}\left(A_2^{\Gamma_j}\right)}(1 + r_{\varepsilon}\rho) \,,
\end{aligned}$$

so we can conclude that

$$\sup_{\substack{1 \leq j \leq N \\ \rho \in [r_{\varepsilon}, r_0]}} \rho^{2m-\tau} \left\| \mathbb{L}_g \tilde{H}^{o}_{\mathbf{hk}}(\rho \cdot) \right\|_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)} \leq C(g) \left\| \mathbf{h}^{\dagger}, \mathbf{k}^{\dagger} \right\|_{\mathcal{B}_{\alpha}} r_{\varepsilon}^{2m-(2+\tau)}$$

and the lemma follows. □

We now give an estimate of the quantity

$$\left\| \mathcal{E}_{r_{\varepsilon}}\left(-\mathbb{L}_g \left(\mathbb{H}^{b}_{\mathbf{hk}} - \varepsilon^{2m} H^{b}_{\mathbf{hk}} \right) + Q_g \left(\mathbb{H}^{b}_{\mathbf{hk}} \right) \right) \right\|_{C^{0,\alpha}_{\tau-2m}(M_{\mathbf{p}})} \,.$$

Lemma 3.4. *Let $\alpha \in (0,1)$, $(\boldsymbol{h}, \boldsymbol{k}) \in \mathfrak{B}(\kappa, \beta, \sigma)$ Then the followng estimate holds*

$$\left\| \mathcal{E}_{r_{\varepsilon}}\left(-\mathbb{L}_g \left(\mathbb{H}^{b}_{\boldsymbol{hk}} - \varepsilon^{2m} H^{b}_{\boldsymbol{hk}} \right) + Q_g \left(\mathbb{H}^{b}_{\boldsymbol{hk}} \right) \right) \right\|_{C^{0,\alpha}_{\tau-2m}(M_{\boldsymbol{p}})} \leq C \varepsilon^{2m+2} r_{\varepsilon}^{-(2+\tau)} \,.$$

Proof. We note that, on M_{r_0}, using estimate (1.8) of Lemma 1.8 we have

$$\begin{aligned}
-\mathbb{L}_g \left(\mathbb{H}^{b}_{\mathbf{hk}} - \varepsilon^{2m} H^{b}_{\mathbf{hk}} \right) + Q_g \left(\mathbb{H}^{b}_{\mathbf{hk}} \right) =& \varepsilon^{2} \bar{b}_j^{2} \mathbb{L}_g \chi_{j,r_0} \psi_{\eta_j}\left(\frac{z}{\bar{b}_j \varepsilon} \right) + Q_g \left(\mathbb{H}^{b}_{\mathbf{hk}} \right) \\
=& \varepsilon^{2m+2} \mathcal{O}_{C^{0,\alpha}\left(M_{r_0}\right)}(1)
\end{aligned}$$

so we have

$$\left\| -\mathbb{L}_g \left(\mathbb{H}^{b}_{\mathbf{hk}} - \varepsilon^{2m} H^{b}_{\mathbf{hk}} \right) + Q_g \left(\mathbb{H}^{b}_{\mathbf{hk}} \right) \right\|_{C^{0,\alpha}\left(M_{r_0}\right)} \leq C(g, \eta_j)\, \varepsilon^{2m+2} \,.$$

Now we estimate the quantity

$$\sup_{\rho \in [r_{\varepsilon}, r_0]} \rho^{2m-\tau} \left\| -\mathbb{L}_g \left(\mathbb{H}^{b}_{\mathbf{hk}} - \varepsilon^{2m} H^{b}_{\mathbf{hk}} \right)(\rho \cdot) + Q_g \left(\mathbb{H}^{b}_{\mathbf{hk}} \right)(\rho \cdot) \right\|_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)} \,.$$

On B_{2r_0}, using Lemma 1.3, we have

$$
\begin{aligned}
-\mathbb{L}_g\left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m} H^b_{\mathbf{hk}}\right) + Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right) =& -\varepsilon^2 \bar{b}_j^2 \mathbb{L}_g\left(\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right) + Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right)\\
=& -\frac{1}{2}\Delta^2\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m}\right)\\
&-\varepsilon^2 \bar{b}_j^2 \mathbb{L}_g\left(\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+Q_g\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right) - Q_g\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
\\
=& -\frac{1}{2}\Delta^2\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&-\varepsilon^2 \bar{b}_j^2 \widetilde{\mathbb{L}}_g\left(\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+Q_0\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+\tilde{Q}_g^2\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right) - Q_g\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
\\
=&\mathbf{s}_0\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&-\varepsilon^2 \bar{b}_j^2 \widetilde{\mathbb{L}}_g\left(\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+\tilde{Q}_g^2\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right) - Q_g\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
\\
=& -\varepsilon^2 \bar{b}_j^2 \widetilde{\mathbb{L}}_g\left(\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+\tilde{Q}_g^2\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right) - Q_g\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right).
\end{aligned}
$$

In the last lines we can see we have the algebraic cancellation

$$
\mathbf{s}_0\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right) = 0
$$

due to the fact that the above term is the scalar curvature of the space X_j. This cancellation let us to have better estimates on $f^o_{\mathbf{b},\mathbf{hk}}$ indeed, if we hadn't used this trick we would not have sufficient control on $f^o_{\mathbf{b},\mathbf{hk}}$ to perform the "Data matching".

We use again Lemma 1.3 and estimate (1.8) on ψ_{η_j} we have

$$\bar{b}_j^2 \varepsilon^2 \widetilde{\mathbb{L}}_g \psi_{\eta_j} \left(\frac{\rho w}{\bar{b}_j \varepsilon} \right) = \varepsilon^{2m+2} \rho^{-2m-2} \mathcal{O}_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)}(1)$$

and then

$$\sup_{\substack{1 \leq j \leq N \\ \rho \in [r_\varepsilon, r_0]}} \rho^{2m-\tau} \left\| \bar{b}_j^2 \varepsilon^2 \widetilde{\mathbb{L}}_g \psi_{\eta_j} \left(\frac{\rho}{\bar{b}_j \varepsilon} \cdot \right) \right\|_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)} \leq C(g, \boldsymbol{\eta}) \, \varepsilon^{2m+2} r_\varepsilon^{-(2+\tau)} .$$

Using again lemma1.3 and the formula (3.1) defining $\mathbb{H}^b_{\mathbf{hk}}$ we find that

$$\tilde{Q}_g^2 \left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m} H^b_{\mathbf{hk}} \right)(\rho w) = \varepsilon^{4m} \rho^{-4m} \mathcal{O}_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)}(1)$$

and so

$$\sup_{\substack{1 \leq j \leq N \\ \rho \in [r_\varepsilon, r_0]}} \rho^{2m-\tau} \left\| \tilde{Q}_g^2 \left(\mathbb{H}^b_{\mathbf{hk}} \right)(\rho \cdot) \right\|_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)} \leq C(g, \boldsymbol{\eta}) \, \varepsilon^{4m} r_\varepsilon^{-(2m+\tau)} .$$

Again, using the shape of $\mathbf{s}_0$ we have

$$\begin{aligned} \mathbf{s}_0 \left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m} H^b_{\mathbf{hk}} \right) - \mathbf{s}_0 \left(-c_{\Gamma_j} \varepsilon^{2m} \bar{b}_j^{2m} |z|^{2-2m} + \varepsilon^2 \bar{b}_j^2 \psi_{\eta_j} \left(\frac{z}{\bar{b}_j \varepsilon} \right) \right) =& \varepsilon^{4m} \rho^{-4m} \mathcal{O}_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)}(1) \\ &+ \varepsilon^{2m} r_\varepsilon^{2m-4+\beta} \rho^{-4m} \mathcal{O}_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)}(1) \end{aligned}$$

so we can conclude that

$$\sup_{\substack{1 \leq j \leq N \\ \rho \in [r_\varepsilon, r_0]}} \rho^{2m-\tau} \left\| Q_0^{[2]} \left(\mathbb{H}^b_{\mathbf{hk}} \right)(\rho w) - Q_0^{[2]} \left(\mathbb{H}^b_{\mathbf{hk}} - \varepsilon^{2m} H^b_{\mathbf{hk}} \right)(\rho w) \right\|_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)} \leq C(g, \boldsymbol{\eta}) \, \varepsilon^{4m} r_\varepsilon^{-(2m+\tau)} .$$

□

From now on we'll consider $f \in C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}}) \oplus \mathcal{D}$ s.t.

$$\|f\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}}) \oplus \mathcal{D}} \leq 2 \left\| \mathcal{E}_{r_\varepsilon} \left(\mathbb{L}_g \mathbb{H}^b_{\mathbf{hk}} - Q_g \left(\mathbb{H}^b_{\mathbf{hk}} \right) \right) \right\|_{C^{0,\alpha}_{\tau-2m}(M_{\mathbf{p}})} \leq \overline{C}(g, \boldsymbol{\eta}) \, \varepsilon^{2m+2} r_\varepsilon^{-(2+\tau)} .$$

Remark 3.3. We remark the fact that the subset of $C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}}) \oplus \mathcal{D}$

$$\|f\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}}) \oplus \mathcal{D}} \leq \overline{C}(g, \boldsymbol{\eta}) \, \varepsilon^{2m+2} r_\varepsilon^{-(2+\tau)}$$

has no dependence on $(\mathbf{h}, \mathbf{k})$ but only on background data as metrics g and η_j.

We set moreover

$$r_\varepsilon^\mu = \varepsilon^{2m+2} r_\varepsilon^{-(2+\tau)} .$$

Lemma 3.5. *Let $\alpha \in (0,1)$. Let $(\boldsymbol{h},\boldsymbol{k}) \in \mathfrak{B}(\kappa,\beta,\sigma)$, let $f \in C^{4,\alpha}_{4+\tau-2m}(M_{\boldsymbol{p}}) \oplus \mathcal{D}$ such that*

$$\|f\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\boldsymbol{p}})\oplus\mathcal{D}} \leq \overline{C}(g,\boldsymbol{\eta})\, r_\varepsilon^{\mu}\,.$$

Then the followng estimate holds

$$\left\|\mathcal{E}_{r_\varepsilon}\left[Q_g\left(\mathbb{H}^b_{\boldsymbol{hk}} + \tilde{H}^o_{\boldsymbol{hk}} + f^o_{\boldsymbol{b},\boldsymbol{hk}}\right) - Q_g\left(\mathbb{H}^b_{\boldsymbol{hk}}\right)\right]\right\|_{C^{0,\alpha}_{\tau-2m}(M_p)} \leq C(g,\boldsymbol{\eta})\,\varepsilon^{2m} r_\varepsilon^{\mu-(2m+2+\tau)}\,.$$

Proof. Using Lemma 1.2 we can see easily that

$$Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right) - Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right) = \left(\varepsilon^{2m} r_\varepsilon^{2m-2+\sigma} + \varepsilon^{2m} r_\varepsilon^{\mu} + r_\varepsilon^{4m-4+2\sigma}\right)\mathcal{O}_{C^{0,\alpha}\left(M_{r_0}\right)}\,(1)$$

and so

$$\begin{aligned}\left\|Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right) - Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right)\right\|_{C^{0,\alpha}\left(M_{r_0}\right)} \leq& C(g,\boldsymbol{\eta})\,\varepsilon^{2m} r_\varepsilon^{\mu}\\ &+ C(g,\boldsymbol{\eta})\,\kappa\left(\varepsilon^{2m} r_\varepsilon^{2m-2+\sigma} + \kappa r_\varepsilon^{4m-4+2\sigma}\right)\,.\end{aligned}$$

We compute now the weighted part of the norm. Using again Lemma 1.3, on $B_{2\rho}(p_j)\backslash B_\rho(p_j)$ we have the following expansion

$$\begin{aligned}Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right) - Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right) =& Q^2_0\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right) + \tilde{Q}^2_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right)\\ &- Q^2_0\left(\mathbb{H}^b_{\mathbf{hk}}\right) - \tilde{Q}^2_g\left(\mathbb{H}^b_{\mathbf{hk}}\right)\\ =& Q^2_0\left(-c_{\Gamma_j}\hat{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\hat{b}_j^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)\right)\\ &+ Q^2_0\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right)\\ &- Q^2_0\left(-c_{\Gamma_j}\hat{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\hat{b}_j^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)\right)\\ &+ \tilde{Q}^2_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}}\right)\\ &- Q^2_0\left(\mathbb{H}^b_{\mathbf{hk}}\right)\\ &+ Q^2_0\left(-c_{\Gamma_j}\overline{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\overline{b}_j^2\psi_{\eta_j}\left(\frac{z}{\overline{b}_j\varepsilon}\right)\right)\\ &- \tilde{Q}^2_g\left(\mathbb{H}^b_{\mathbf{hk}}\right)\\ &+ Q^2_0\left(-c_{\Gamma_j}\overline{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m} + \varepsilon^2\overline{b}_j^2\psi_{\eta_j}\left(\frac{z}{\overline{b}_j\varepsilon}\right)\right)\,.\end{aligned}$$

Adding and subtracting the same quantities we have

$$
\begin{aligned}
Q_g\left(\mathbb{H}^b_{\mathbf{hk}}+\tilde{H}^o_{\mathbf{hk}}+f^o_{\mathbf{b},\mathbf{hk}}\right)-Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right)=&-\frac{1}{2}\Delta^2\left(-c_{\Gamma_j}\hat{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m}+\varepsilon^2\hat{b}_j^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)\right)\\
&+Q_0^2\left(-c_{\Gamma_j}\hat{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m}+\varepsilon^2\hat{b}_j^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)\right)\\
&+Q_0^2\left(\mathbb{H}^b_{\mathbf{hk}}+\tilde{H}^o_{\mathbf{hk}}+f^o_{\mathbf{b},\mathbf{hk}}\right)\\
&-Q_0^2\left(-c_{\Gamma_j}\hat{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m}+\varepsilon^2\hat{b}_j^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)\right)\\
&+\tilde{Q}_g^2\left(\mathbb{H}^b_{\mathbf{hk}}+\tilde{H}^o_{\mathbf{hk}}+f^o_{\mathbf{b},\mathbf{hk}}\right)-\tilde{Q}_g^2\left(\mathbb{H}^b_{\mathbf{hk}}\right)\\
&-Q_0^2\left(\mathbb{H}^b_{\mathbf{hk}}\right)\\
&+Q_0^2\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m}+\varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+\frac{1}{2}\Delta^2\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m}+\varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&-Q_0^2\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m}+\varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+\frac{1}{2}\Delta^2\left(\varepsilon^2\hat{b}_j^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)-\varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
=&Q_0^2\left(\mathbb{H}^b_{\mathbf{hk}}+\tilde{H}^o_{\mathbf{hk}}+f^o_{\mathbf{b},\mathbf{hk}}\right)\\
&-Q_0^2\left(-c_{\Gamma_j}\hat{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m}+\varepsilon^2\hat{b}_j^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)\right)\\
&+\tilde{Q}_g^2\left(\mathbb{H}^b_{\mathbf{hk}}+\tilde{H}^o_{\mathbf{hk}}+f^o_{\mathbf{b},\mathbf{hk}}\right)-\tilde{Q}_g^2\left(\mathbb{H}^b_{\mathbf{hk}}\right)\\
&-Q_0^2\left(\mathbb{H}^b_{\mathbf{hk}}\right)\\
&+Q_0^2\left(-c_{\Gamma_j}\bar{b}_j^{2m}\varepsilon^{2m}|z|^{2-2m}+\varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right)\\
&+\frac{1}{2}\Delta^2\left(\varepsilon^2\hat{b}_j^2\psi_{\eta_j}\left(\frac{z}{\hat{b}_j\varepsilon}\right)-\varepsilon^2\bar{b}_j^2\psi_{\eta_j}\left(\frac{z}{\bar{b}_j\varepsilon}\right)\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\sup_{\substack{1\leq j\leq N\\ \rho\in[r_\varepsilon,r_0]}}\rho^{2m-\tau}\left\|Q_g\left(\mathbb{H}^b_{\mathbf{hk}}+\tilde{H}^o_{\mathbf{hk}}+f^o_{\mathbf{b},\mathbf{hk}}\right)-Q_g\left(\mathbb{H}^b_{\mathbf{hk}}\right)\right\|_{C^{0,\alpha}\left(A_2^{\Gamma_j}\right)}\leq&C(g)\,\kappa\varepsilon^{2m}r_\varepsilon^{\sigma-(4+\tau)}\\
&+C(g)\,\varepsilon^{2m}r_\varepsilon^{\mu-(2m+2+\tau)}\\
&+C(g)\,\kappa^2r_\varepsilon^{2m+2\sigma-(6+\tau)}.
\end{aligned}
$$

□

Now we want to estimate the lipschitz constant of operator $\mathcal{N}$.

Lemma 3.6. *Let* $(\boldsymbol{h},\boldsymbol{k}) \in \mathfrak{B}(\kappa,\beta,\sigma)$, $f, f' \in C^{4,\alpha}_{4+\tau-2m}(M_{\boldsymbol{p}}) \oplus \mathcal{D}$ *and*

$$\|f\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}}, \|f'\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}} \leq \overline{C}(g,\boldsymbol{\eta})\, r_\varepsilon^\mu$$

then

$$\|\mathcal{N}(f,\boldsymbol{h},\boldsymbol{k}) - \mathcal{N}(f',\boldsymbol{h},\boldsymbol{k})\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}} \leq \frac{1}{2}\|f-f'\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}}\,.$$

Proof. We recall that

$$\mathcal{N}(f,\mathbf{h},\mathbf{k}) - \mathcal{N}(f',\mathbf{h},\mathbf{k}) = \mathbb{G}_\tau\left[\mathcal{E}_{r_\varepsilon} Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f\right) - \mathcal{E}_{r_\varepsilon} Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f'\right) + \nu'(f) - \nu'(f')\right].$$

So we only need to estimate the quantity

$$\left\|\mathcal{E}_{r_\varepsilon} Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f\right) - \mathcal{E}_{r_\varepsilon} Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f'\right)\right\|_{C^{0,\alpha}_{\tau-2m}(M_{\mathbf{p}})}$$

.

On M_{r_0}, using Lemma 1.2, we have

$$\begin{aligned}\left\|Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f\right) - Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f'\right)\right\|_{C^{4,\alpha}\left(M_{r_0}\right)} \leq & C(g,\kappa,\boldsymbol{\eta})\, r_\varepsilon^{2m-2+\sigma}\|f-f'\|_{C^{4,\alpha}\left(M_{r_0}\right)} \\ & + C(g,\kappa,\boldsymbol{\eta})\,\epsilon^{2m}\|f-f'\|_{C^{4,\alpha}\left(M_{r_0}\right)}\,.\end{aligned}$$

For the weighted part of the norm we have

$$\sup_{\rho\in[r_\varepsilon,r_0]} \rho^{2m-\tau}\left\||Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f\right) - Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f'\right)\right\|_{C^{0,\alpha}(B_2\setminus B_1)} \leq D(\varepsilon)\|f-f'\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})\oplus\mathcal{D}}$$

with

$$D(\varepsilon) = C(g,\boldsymbol{\eta},\kappa)\left(\epsilon^{2m} r_\varepsilon^{-(2m+2+\tau)} + r_\varepsilon^{\sigma-4-\tau} + r_\varepsilon^{\mu-(2m+2+\tau)}\right).$$

By the choice of parameters σ, μ, β, τ, if we choose ε sufficiently small we have the proposition. □

Lemma 3.7. *Let* $f \in C^{4,\alpha}_{4+\tau-2m}(M_{\boldsymbol{p}}) \oplus \mathcal{D}$ *with*

$$\|f\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}} \leq \overline{C}(g,\boldsymbol{\eta})\, r_\varepsilon^\mu\,,$$

then

$$\mathcal{N}(f,\cdot,\cdot) : \mathfrak{B}(\kappa,\beta,\sigma) \to C^{4,\alpha}_{4+\tau-2m}(M_{\boldsymbol{p}}) \oplus \mathcal{D}$$

is lipschiz.

Proof. By Lemma 3.3 we have

$$\begin{aligned}\left\|\mathcal{E}_{r_\varepsilon}\mathbb{L}_g \tilde{H}^o_{\mathbf{hk}} - \mathcal{E}_{r_\varepsilon}\mathbb{L}_g \tilde{H}^o_{\mathbf{h'k'}}\right\|_{C^{0,\alpha}_{\tau-2m}(M_{\mathbf{p}})} = & \left\|\mathcal{E}_{r_\varepsilon}\mathbb{L}_g \tilde{H}^o_{\mathbf{h}-\mathbf{h'},\mathbf{k}-\mathbf{k'}}\right\|_{C^{0,\alpha}_{\tau-2m}(M_{\mathbf{p}})} \\ \leq & C(g,\boldsymbol{\eta})\, r_\varepsilon^{2m-(2+\tau)}\left\|\mathbf{h}-\mathbf{h'},\mathbf{k}-\mathbf{k'}\right\|.\end{aligned}$$

Doing computations completely analogous to Lemma 3.4 and Lemma 3.6 we get

$$\left\|\mathcal{E}_{r_\varepsilon} Q_g\left(\mathbb{H}^b_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f\right) - \mathcal{E}_{r_\varepsilon} Q_g\left(\mathbb{H}^b_{\mathbf{h'k'}} + \tilde{H}^o_{\mathbf{h'k'}} + f\right)\right\|_{C^{0,\alpha}_{\tau-2m}(M_{\mathbf{p}})} \leq D(\varepsilon)\left\|\mathbf{h}-\mathbf{h}', \mathbf{k}-\mathbf{k}'\right\|_{\mathcal{B}_\alpha}$$

with

$$D(\varepsilon) = C(g, \boldsymbol{\eta})\,\varepsilon^{2m} r_\varepsilon^{-4-\tau}$$

and so the lemma follows.

□

We can summarize the properties of the operator $\mathcal{N}$ in the following result.

Proposition 3.3. *Let* $(\boldsymbol{h}, \boldsymbol{k}), (\boldsymbol{h}', \boldsymbol{k}') \in \mathfrak{B}(\kappa, \beta, \sigma)$, $f, f' \in C^{4,\alpha}_{4+\tau-2m}(M_{\boldsymbol{p}}) \oplus \mathcal{D}$ *such that*

$$\|f\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}}, \|f'\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}} \leq \overline{C}(g, \boldsymbol{\eta})\, r_\varepsilon^\mu,$$

then

1. $\|\mathcal{N}(f, \boldsymbol{h}, \boldsymbol{k})\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}} \leq C(g, \eta_i)\, r_\varepsilon^\mu$,
2. $\|\mathcal{N}(f, \boldsymbol{h}, \boldsymbol{k}) - \mathcal{N}(f', \boldsymbol{h}, \boldsymbol{k})\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}} \leq \frac{1}{2}\|f - f'\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}}$,
3. $\left\|\mathcal{N}(f, \boldsymbol{h}, \boldsymbol{k}) - \mathcal{N}(f, \boldsymbol{h}', \boldsymbol{k}')\right\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}} \leq \frac{1}{2}\left\|\boldsymbol{h}-\boldsymbol{h}', \boldsymbol{k}-\boldsymbol{k}'\right\|_{\mathcal{B}_\alpha}$.

Proof. 1. Follows immediately combining Lemmas 3.3, 3.4, 3.5

2. Is Lemma 3.6

3. Is Lemma 3.7

□

Now, finally, we can construct our family of CscK metrics $g_{\mathbf{b},\mathbf{hk}}$ on M_{r_ε}.

Proposition 3.4. *Let* $(\boldsymbol{h}, \boldsymbol{k}) \in \mathfrak{B}(\kappa, \beta, \sigma)$ *then there exist* $f^o_{\boldsymbol{b},\boldsymbol{hk}} \in C^{4,\alpha}_{4+\tau-2m}(M_{\boldsymbol{p}}) \oplus \mathcal{D}$ *such that*

$$\left\|f^o_{\boldsymbol{b},\boldsymbol{hk}}\right\|_{C^{4,\alpha}_{4+\tau-2m}(M_p)\oplus\mathcal{D}} \leq \overline{C}(g, \boldsymbol{\eta})\, r_\varepsilon^\mu$$

and on M_{r_ε}

$$\omega_{g_{\boldsymbol{b},\boldsymbol{hk}}} = \omega_g + i\partial\overline{\partial}\mathbb{F}^o_{\boldsymbol{b},\boldsymbol{hk}}$$

is a Kähler metric of constant scalar curvature with

$$\left|s_{g_{\boldsymbol{b},\boldsymbol{hk}}} - s_g\right| \leq C(g, \boldsymbol{\eta})\,\varepsilon^{2m}.$$

Moreover the metric $g_{\boldsymbol{b},\boldsymbol{hk}}$ *depends continuosly on* $(\boldsymbol{h}, \boldsymbol{k}) \in \mathfrak{B}(\kappa, \beta, \sigma)$.

Proof. By Proposition 3.3 we know that for fixed $(\mathbf{h}, \mathbf{k}) \in \mathfrak{B}(\kappa, \beta, \sigma)$ the operator

$$\mathcal{N}(\cdot, \mathbf{h}, \mathbf{k}) : C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}}) \oplus \mathcal{D} \to C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}}) \oplus \mathcal{D}$$

is contractive on the subset of $C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}}) \oplus \mathcal{D}$

$$\|f\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})\oplus\mathcal{D}} \leq \overline{C}(g, \boldsymbol{\eta})\, r_\varepsilon^\mu.$$

So by contraction theorem we have a unique fixed point $f^o_{\mathbf{b},\mathbf{hk}} \in C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}}) \oplus \mathcal{D}$ with

$$\left\| f^o_{\mathbf{b},\mathbf{hk}} \right\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})\oplus\mathcal{D}} \leq \overline{C}\,(g,\boldsymbol{\eta})\, r_\varepsilon^\mu$$

that by construction has the properties stated in the proposition. The only nontrivial feature is the continuity of $g_{\mathbf{b},\mathbf{hk}}$ with respect to $(\mathbf{h},\mathbf{k}) \in \mathfrak{B}(\kappa,\beta,\sigma)$. To prove this property it is sufficient to prove that $f^o_{\mathbf{b},\mathbf{hk}}$ depend contiunuously on $(\mathbf{h},\mathbf{k}) \in \mathfrak{B}(\kappa,\beta,\sigma)$.

$$\begin{aligned}
\left\| f^o_{\mathbf{b},\mathbf{hk}} - f^o_{\mathbf{b},\mathbf{h}'\mathbf{k}'} \right\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})\oplus\mathcal{D}} &= \left\| \mathcal{N}\left(f^o_{\mathbf{b},\mathbf{hk}},\mathbf{h},\mathbf{k}\right) - \mathcal{N}\left(f^o_{\mathbf{b},\mathbf{h}'\mathbf{k}'},\mathbf{h}',\mathbf{k}'\right) \right\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})\oplus\mathcal{D}}\\
&\leq \left\| \mathcal{N}\left(f^o_{\mathbf{b},\mathbf{hk}},\mathbf{h},\mathbf{k}\right) - \mathcal{N}\left(f^o_{\mathbf{b},\mathbf{h}'\mathbf{k}'},\mathbf{h},\mathbf{k}\right) \right\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})\oplus\mathcal{D}}\\
&\quad + \left\| \mathcal{N}\left(f^o_{\mathbf{b},\mathbf{h}'\mathbf{k}'},\mathbf{h},\mathbf{k}\right) - \mathcal{N}\left(f^o_{\mathbf{b},\mathbf{h}'\mathbf{k}'},\mathbf{h}',\mathbf{k}'\right) \right\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})\oplus\mathcal{D}}\\
&\leq \frac{1}{2} \left\| f^o_{\mathbf{b},\mathbf{hk}} - f^o_{\mathbf{b},\mathbf{h}'\mathbf{k}'} \right\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})\oplus\mathcal{D}}\\
&\quad + C\,(g,\boldsymbol{\eta},\varepsilon) \left\| \mathbf{h}-\mathbf{h}',\mathbf{k}-\mathbf{k}' \right\|_{\mathcal{B}_\alpha}
\end{aligned}$$

and so we can conclude that

$$\left\| f^o_{\mathbf{b},\mathbf{hk}} - f^o_{\mathbf{b},\mathbf{h}'\mathbf{k}'} \right\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\mathbf{p}})\oplus\mathcal{D}} \leq \left\| \mathbf{h}-\mathbf{h}',\mathbf{k}-\mathbf{k}' \right\|_{\mathcal{B}_\alpha}\,.$$

□

3.2 Construction of the families of metrics on ALE spaces

Now we want to prepare the model spaces X_j's for the gluing: we will follow steps 12 to 15. We want, indeed, to construct families of Kähler metrics that are cscK on big compact sets and that depend on suitable parameters. As in the base cscK orbifold we will create this family perturbing the rescaled background metric $\hat{b}_j^2\varepsilon^{2m}\omega_{\eta_j}$ with $\varepsilon^2\partial\overline{\partial}\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ with $\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ a sufficiently regular function that we want to build carefully. We want to construct a functions $\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j} \in C^{4,\alpha}\left(X_{\frac{R_\varepsilon}{\hat{b}_j},j}\right)$ with the following features:

1. on a large annulus $X_{\frac{R_\varepsilon}{\hat{b}_j},j} \setminus X_{\frac{R_\varepsilon}{2\hat{b}_j},j}$ have to be very close to the (suitably rescaled) potential ψ_g of g in small neighborhoods of p_j;
2. have to depend on their behavior at $\partial X_{\frac{R_\varepsilon}{\hat{b}_j},j}$;
3. the resulting metrics must constant scalar curvature on $X_{\frac{R_\varepsilon}{\hat{b}_j},j}$.

To satisfy these requests, as in the base manifold case we build $\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ by blocks:

$$\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j} = \mathbb{J}_{\hat{b}} + \tilde{H}^I_{\tilde{h}_j\tilde{k}_j} + f^I_{\tilde{h}_j\tilde{k}_j}$$

and in particular

1. $\mathbb{J}_{\hat{b}}$ is the "skeleton" and will satisfy the first request;
2. $\tilde{H}^I_{\tilde{h}_j\tilde{k}_j}$ will prescribe the behavior at $\partial X_{\frac{R_\varepsilon}{\hat{b}_j},j}$;

3. $f^{I}_{\tilde{h}_j\tilde{k}_j}$ will be the correction term assuring that the scalar curvature is constant.

For the sake of notation, in the rest of the section we will drop everywhere the subscript j relative to the point p_j. We also recall that in section 1.6 there is the guide line to the proof of Theorem 1.7 and in section 1.7 there are notations and definitions of cutoff functions we will use.

3.2.1 Construction of the skeleton $\mathbb{J}_{\hat{b}}$

From now on (X,η) will be a Ricci-flat ALE Kähler manifold. The condition $\mathrm{Ric}\,(\eta)=0$ implies that

$$\mathbb{L}_\eta = \frac{\Delta_\eta^2}{2}\,.$$

Let ψ_g the local potential of the metric g on a small neighborhood of $p\in M$, we recall that by Remark 1.2 we can expand ψ_g in the following way

$$\psi_g(z)=\sum_{k=0}^{+\infty}p_{4+k}\left(z\right)\,.$$

We consider the functions

$$\chi_{R_0}p_4,\chi_{R_0}p_5\in C^\infty_{loc}\left(X\right)\,.$$

We want to modify these functions in such a way that they are "almost" in $\ker\left(\Delta_\eta^2\right)$. We start with $\chi_{R_0}p_4$ and calculate

$$\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}p_4\right)\,.$$

By Remark 1.2 equation (1.5), on $X\setminus X_{\frac{R_0}{\hat{b}}}$, we have that

$$\begin{aligned}\Delta_\eta p_4 =&\Delta p_4+8c_\Gamma\partial_j\overline{\partial_i}|x|^{2-2m}\partial_i\overline{\partial_j}p_4+\mathcal{O}\left(|x|^{-2m}\right)\\ =&\Delta p_4+8c_\Gamma\left[\frac{(1-m)}{|x|^{2m}}\delta_{j\bar{i}}+\frac{(m-1)\,m}{|x|^{2m+2}}\overline{x^i}x^j\right]\partial_i\overline{\partial_j}p_4+\mathcal{O}\left(|x|^{-2m}\right)\\ =&\Delta p_4+8c_\Gamma\left[\frac{(1-m)}{4|x|^{2m}}\Delta p_4+\frac{4\,(m-1)\,m}{|x|^{2m+2}}p_4\right]+\mathcal{O}\left(|x|^{-2m}\right)\end{aligned}$$

and then

$$
\begin{aligned}
\frac{1}{2}\Delta_\eta^2 p_4 &= \frac{\Delta_\eta}{2}\left(\Delta_\eta p_4\right) \\
&= \frac{\Delta_\eta}{2}\left[\Delta p_4 - \frac{2c_\Gamma\left(m-1\right)}{|x|^{2m}}\Delta p_4 + \frac{32c_\Gamma\left(m-1\right)m}{|x|^{2m+2}}p_4 + 4\theta'_{i\bar{j}}\partial_j\overline{\partial}_i p_4\right] \\
&= \frac{1}{2}\left[\Delta^2 p_4 - \frac{2c_\Gamma\left(m-1\right)}{|x|^{2m}}\Delta^2 p_4 + \frac{8c_\Gamma\left(m-1\right)m}{|x|^{2m+2}}\overline{x^i}x^j\partial_j\overline{\partial}_i\Delta p_4\right] \\
&+ \frac{\Delta}{2}\left[-\frac{2c_\Gamma\left(m-1\right)}{|x|^{2m}}\Delta p_4 + \frac{32c_\Gamma\left(m-1\right)m}{|x|^{2m+2}}p_4\right] + \mathcal{O}\left(|x|^{-2m-2}\right) \\
&= \frac{1}{2}\left[-2s_g + \frac{4c_\Gamma\left(m-1\right)}{|x|^{2m}}s_g + \frac{8c_\Gamma\left(m-1\right)m}{|x|^{2m}}\left(-\frac{s_g}{2m} + \tilde{\phi}_2\right)\right] \\
&+ \frac{\Delta}{2}\left[\frac{2c_\Gamma\left(m-1\right)}{|x|^{2m-2}}\left(-\frac{s_g\left(m-1\right)}{2m\left(m+1\right)} + \tilde{\phi}_2 + \tilde{\phi}_4\right)\right] + \mathcal{O}\left(|x|^{-2m-2}\right) \\
&= -s_g - \frac{1}{|x|^{2m}}\tilde{\phi}_2 - \frac{1}{|x|^{2m}}\tilde{\phi}_4 + \mathcal{O}\left(|x|^{-2m-2}\right)\,.
\end{aligned}
$$

Now let

$$
\tilde{u}_4 = c_2\chi_{R_0}|x|^{4-2m}\tilde{\phi}_2 + c_4\chi_{R_0}|x|^{4-2m}\tilde{\phi}_4
$$

with $c_2, c_4 \in \mathbb{R}$ such that

$$
\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}p_4 - \tilde{u}_4\right) + s_g = \mathcal{O}\left(|x|^{-2-2m}\right)\,.
$$

So we have

$$
\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}p_4 - \tilde{u}_4\right) + s_g \in C^{0,\alpha}_{-2-2m+\delta'}\left(X\right) \qquad \text{with } \delta' \in (0,1)\,.
$$

We would like to solve the equation

$$
\frac{1}{2}\Delta_\eta^2\bar{u}_4 = \frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}p_4 - \tilde{u}_4\right) + s_g
$$

with $\bar{u}_4 \in C^{0,\alpha}_{2-2m+\delta'}\left(X\right)$ but Proposition 2.6 tells us that we can if and only if

$$
\int_X\left[\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}p_4 - \tilde{u}_4\right) + s_g\right]d\mu_\eta = 0\,.
$$

We now compute the integral

$$
\int_X\left[\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}p_4\right) + s_g\right]d\mu_\eta\,.
$$

Applying divergence theorem and Lemma 2.2 we have

$$\begin{aligned}\int_X \left[\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}p_4\right)+s_g\right]d\mu_\eta &= \lim_{\rho\to+\infty}\int_{X_\rho}\left[\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}p_4\right)+s_g\right]d\mu_\eta\\
&=\lim_{\rho\to+\infty}\left[\frac{1}{2}\int_{X_\rho}\Delta_\eta^2\left(\chi_{R_0}p_4\right)d\mu_\eta+\int_{X_\rho}s_g d\mu_\eta\right]\\
&=\lim_{\rho\to+\infty}\left[\frac{1}{2}\int_{\partial X_\rho}\partial_\nu\Delta_\eta\left(\chi_{R_0}p_4\right)d\mu_\eta+\int_{X_\rho}s_g d\mu_\eta\right]\\
&=\lim_{\rho\to+\infty}\left[\frac{1}{2}\int_{\partial X_\rho}\partial_\nu\Delta_\eta\left(\chi_{R_0}p_4\right)d\mu_\eta+\frac{s_g\mu\left(S^{2m-1}\right)}{2m\left|\Gamma\right|}\rho^{2m}\right].\end{aligned}$$

By Lemma 2.2 we have that

$$d\mu_\eta|_{\partial X_\rho}=\nu\lrcorner\, d\mu_0|_{\partial X_\rho}\ ,$$

$$\begin{aligned}\nu\lrcorner d\mu_\eta =&\nu\lrcorner d\mu_0\\
=&\left(\frac{\left(x^i\partial_i+\overline{x^i}\overline{\partial}_i\right)}{|x|}\left[1+\frac{c_\Gamma\left(m-1\right)^2}{|x|^{2m}}+\mathcal{O}\left(|x|^{-1-2m}\right)\right]\right)\lrcorner d\mu_0\\
=&\left[1+\frac{c_\Gamma\left(m-1\right)^2}{|x|^{2m}}\right]\frac{\left(x^i\partial_i+\overline{x^i}\overline{\partial}_i\right)}{|x|}\lrcorner d\mu_0+\mathcal{O}\left(|x|^{-1-2m}\right)\lrcorner d\mu_0\,,\end{aligned}$$

so we have

$$d\mu_\eta|_{\partial X_\rho}=\left[1+\frac{c_\Gamma\left(m-1\right)^2}{|x|^{2m}}+\mathcal{O}\left(|x|^{-2m-1}\right)\right]|x|^{2m-1}\ d\mu_0|_{\partial B_1/\Gamma}\ .$$

Now we evaluate $\partial_\nu\Delta_\eta p_4$.

$$\begin{aligned}\partial_\nu\Delta_\eta p_4 =&\partial_\nu\left[\Delta p_4+8c_\Gamma\left(\frac{(1-m)}{4|x|^{2m}}\Delta p_4+\frac{4\left(m-1\right)m}{|x|^{2m+2}}p_4\right)+4\theta^{'}_{i\bar{j}}\partial_j\overline{\partial}_i p_4\right]\\
=&\partial_\rho\Delta p_4+\frac{c_\Gamma\left(m-1\right)^2}{|x|^{2m}}\partial_\rho\Delta p_4\\
&+8c_\Gamma\partial_\rho\left(\frac{(1-m)}{4|x|^{2m}}\Delta p_4+\frac{4\left(m-1\right)m}{|x|^{2m+2}}p_4\right)+\mathcal{O}\left(|x|^{-2m}\right)\\
=&\partial_\rho\Delta p_4+\frac{c_\Gamma\left(m-1\right)^2}{|x|^{2m}}\partial_\rho\Delta p_4+2c_\Gamma\left(1-m\right)\partial_\rho\left(\frac{\Delta p_4}{|x|^{2m}}\right)\\
&+32c_\Gamma\left(m-1\right)m\partial_\rho\left(\frac{p_4}{|x|^{2m+2}}\right)+\mathcal{O}\left(|x|^{-2m}\right)\ .\end{aligned}$$

We recall that p_4 has the form

$$p_4=\alpha\rho^4+\tilde{\phi}_2\rho^4+\tilde{\phi}_4\rho^4\,,$$

$$\begin{aligned}
\partial_\nu \Delta_\eta p_4 =& \partial_\rho \left(8\left(m+1\right)\alpha\rho^2 + \tilde{\phi}_2\rho^2\right) + \frac{c_\Gamma\left(m-1\right)^2}{\rho^{2m}}\partial_\rho\left(8\left(m+1\right)\alpha\rho^2 + \tilde{\phi}_2\rho^2\right) \\
&+ 2c_\Gamma\left(1-m\right)\partial_\rho\left(\frac{8\left(m+1\right)\alpha\rho^2 + \tilde{\phi}_2'\rho^2}{\rho^{2m}}\right) \\
&+ 32c_\Gamma\left(m-1\right)m\partial_\rho\left(\frac{\alpha\rho^4 + \tilde{\phi}_2\rho^4 + \tilde{\phi}_4\rho^4}{\rho^{2m+2}}\right) + \mathcal{O}\left(|x|^{-2m}\right) \\
=& 16\left(m+1\right)\alpha\rho + \frac{c_\Gamma\left(m-1\right)^2}{\rho^{2m}}\left(16\left(m+1\right)\alpha\rho\right) + 2c_\Gamma\left(1-m\right)\partial_\rho\left(\frac{8\left(m+1\right)\alpha}{\rho^{2m-2}}\right) \\
&+ 32c_\Gamma\left(m-1\right)m\partial_\rho\left(\frac{\alpha}{\rho^{2m-2}}\right) + \mathcal{O}\left(|x|^{1-2m}\right)\left(\tilde{\phi}_2+\tilde{\phi}_4\right) + \mathcal{O}\left(|x|^{-2m}\right) \\
=& 16\left(m+1\right)\alpha\rho + \frac{16\left(m+1\right)c_\Gamma\left(m-1\right)^2\alpha}{\rho^{2m-1}} + \frac{32\left(m+1\right)c_\Gamma\left(m-1\right)^2\alpha}{\rho^{2m-1}} \\
&- \frac{64c_\Gamma\left(m-1\right)^2 m\alpha}{\rho^{2m-1}} + \mathcal{O}\left(|x|^{1-2m}\right)\left(\tilde{\phi}_2+\tilde{\phi}_4\right) + \mathcal{O}\left(|x|^{-2m}\right) \\
=& 16\left(m+1\right)\alpha\rho + \frac{16\left(m+1\right)c_\Gamma\left(m-1\right)^2\alpha}{\rho^{2m-1}} - \frac{32c_\Gamma\left(m-1\right)^3\alpha}{\rho^{2m-1}} + \mathcal{O}\left(|x|^{1-2m}\right)\left(\tilde{\phi}_2+\tilde{\phi}_4\right) + \mathcal{O}\left(|x|^{-2m}\right) \\
=& 16\left(m+1\right)\alpha\rho - \frac{16\left(m-3\right)\left(m-1\right)^2c_\Gamma\alpha}{\rho^{2m-1}} + \mathcal{O}\left(|x|^{1-2m}\right)\left(\tilde{\phi}_2+\tilde{\phi}_4\right) + \mathcal{O}\left(|x|^{-2m}\right)\ ,
\end{aligned}$$

$$\begin{aligned}
\partial_\nu \Delta_\eta p_4\ d\mu_\eta|_{\partial X_\rho} =& \left[16\left(m+1\right)\alpha\rho - \frac{16\left(m-3\right)\left(m-1\right)^2c_\Gamma\alpha}{\rho^{2m-1}}\right]\left[\rho^{2m-1} + \frac{c_\Gamma\left(m-1\right)^2}{\rho}\right] d\mu_0|_{\partial B_1/\Gamma} \\
&+ \left[16\left(m+1\right)\alpha\rho - \frac{16\left(m-3\right)\left(m-1\right)^2c_\Gamma\alpha}{\rho^{2m-1}}\right]\mathcal{O}\left(\frac{1}{\rho^2}\right) d\mu_0|_{\partial B_1/\Gamma} \\
&+ \left[\mathcal{O}\left(\rho^{1-2m}\right)\left(\tilde{\phi}_2+\tilde{\phi}_4\right) + \mathcal{O}\left(\rho^{-2m}\right)\right]\left[\rho^{2m-1} + \frac{c_\Gamma\left(m-1\right)^2}{\rho} + \mathcal{O}\left(\rho^{-2}\right)\right] d\mu_0|_{\partial B_1/\Gamma} \\
=& \left[16\left(m+1\right)\alpha\rho^{2m} + 16\left(m+1\right)c_\Gamma\left(m-1\right)^2\alpha - 16\left(m-3\right)\left(m-1\right)^2c_\Gamma\alpha\right] d\mu_0|_{\partial B_1/\Gamma} \\
&+ \left[\mathcal{O}\left(1\right)\tilde{\phi}_2 + \mathcal{O}\left(1\right)\tilde{\phi}_4 + \mathcal{O}\left(\rho^{-1}\right)\right] d\mu_0|_{\partial B_1/\Gamma} \\
=& \left[16\left(m+1\right)\alpha\rho^{2m} + 64c_\Gamma\left(m-1\right)^2\alpha + \mathcal{O}\left(1\right)\left(\tilde{\phi}_2+\tilde{\phi}_4\right) + \mathcal{O}\left(\rho^{-1}\right)\right] d\mu_0|_{\partial B_1/\Gamma} \\
=& \left[-\frac{s_g}{m}\rho^{2m} - \frac{4c_\Gamma\left(m-1\right)^2 s_g}{m\left(m+1\right)} + \mathcal{O}\left(1\right)\left(\tilde{\phi}_2+\tilde{\phi}_4\right) + \mathcal{O}\left(\rho^{-1}\right)\right] d\mu_0|_{\partial B_1/\Gamma}\ .
\end{aligned}$$

In last lines we used the fact that

$$\alpha = -\frac{s_g}{16m\left(m+1\right)}\,.$$

Finally

$$\begin{aligned}\frac{1}{2}\int_{X_\rho} \partial_\nu \Delta_\eta p_4 d\mu_\eta =& \frac{1}{2}\int_{\partial B_1/\Gamma}\left[-\frac{s_g}{m}\rho^{2m} - \frac{4c_\Gamma\left(m-1\right)^2 s_g}{m\left(m+1\right)} + \mathcal{O}\left(\rho^{-1}\right)\right] d\mu_0|_{\partial B_1/\Gamma}\\ &+ \frac{1}{2}\int_{\partial B_1/\Gamma}\left[\mathcal{O}\left(1\right)\tilde{\phi}_2 + \mathcal{O}\left(1\right)\tilde{\phi}_4 + \mathcal{O}\left(\rho^{-1}\right)\right] d\mu_0|_{\partial B_1/\Gamma}\\ =& -\frac{s_g \mu\left(S^{2m-1}\right)\rho^{2m}}{2m|\Gamma|} - \frac{2c_\Gamma\left(m-1\right)^2 \mu\left(S^{2m-1}\right) s_g}{m\left(m+1\right)|\Gamma|} + \mathcal{O}\left(\rho^{-1}\right)\end{aligned}$$

and so

$$\frac{1}{2}\int_X \left[\Delta_\eta^2 \chi_{R_0} p_4 + 2s_g\right] d\mu_\eta = -\frac{2c_\Gamma\left(m-1\right)^2 \mu\left(S^{2m-1}\right) s_g}{m\left(m+1\right)|\Gamma|}\,.$$

We found that

$$\int_X \left(\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0} p_4 - \tilde{u}_4\right) + s_g\right) d\mu_\eta \neq 0\,.$$

To overcome this difficulty we try solve a slightly different equation:

$$\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0} p_4 - \tilde{u}_4 + A\chi_{R_0}|x|^{4-2m} + \bar{u}_4\right) = -s_g$$

with $A \in \mathbb{R}$ to be determined and $\bar{u}_4 \in C^{4,\alpha}_{2-2m+\delta'}\left(X\right)$. The equation in $\bar{u}_4$ becomes

$$\frac{1}{2}\Delta_\eta^2 \bar{u}_4 = -\left[s_g + \frac{1}{2}\Delta_\eta^2\left(\chi_{R_0} p_4 - \tilde{u}_4\right) + \frac{A}{2}\Delta_\eta^2 \chi_{R_0}|x|^{4-2m}\right]\,.$$

Clearly the right hand side of this equation is in $C^{4,\alpha}_{-2-2m+\delta'}\left(X\right)$. It is immediate to see that

$$\frac{1}{2}\int_X \Delta_\eta^2 \chi_{R_0}|x|^{4-2m} = \frac{4\left(m-2\right)\left(m-1\right)\mu\left(S^{2m-1}\right)}{|\Gamma|}\,,$$

so setting

$$A = \frac{c_\Gamma\left(m-1\right)s_g}{2\left(m-2\right)m\left(m+1\right)}$$

we have that

$$\int_X \left[s_g + \frac{1}{2}\Delta_\eta^2\left(\chi_{R_0} p_4 - \tilde{u}_4\right) + \frac{c_\Gamma\left(m-1\right)s_g}{4\left(m-2\right)m\left(m+1\right)}\Delta_\eta^2 \chi_{R_0}|x|^{4-2m}\right] d\mu_\eta = 0$$

and so by Propositions 2.6 and 2.7 we can find $\bar{u}_4 \in C^{4,\alpha}_{2-2m+\delta'}\left(X\right)$ such that

$$\frac{1}{2}\Delta_\eta^2 \bar{u}_4 = -\left[s_g + \frac{1}{2}\Delta_\eta^2\left(\chi_{R_0} p_4 - \tilde{u}_4\right) + \frac{c_\Gamma\left(m-1\right)s_g}{4\left(m-2\right)m\left(m+1\right)}\Delta_\eta^2 \chi_{R_0}|x|^{4-2m}\right]\,.$$

We define

$$u_4 := \tilde{u}_4 - \bar{u}_4$$

Now we deal with $\chi_{R_0}p_5$. Using equation (1.6) we have

$$\begin{aligned}\frac{1}{2}\Delta^2_\eta\left(\chi_{R_0}p_5\right) =& \frac{1}{2}\Delta^2\left(\chi_{R_0}p_5\right)+\tilde{\mathbb{L}}_\eta\left(\chi_{R_0}p_5\right)\\ =&\tilde{\mathbb{L}}_\eta\left(\chi_{R_0}p_5\right)\\ =&\mathcal{O}\left(|x|^{1-2m}\left(\sum_{h=1}^{K}\phi_{2h+1}\right)\right).\end{aligned}$$

As for p_4 we correct “by hand” p_5 subtracting a term $\tilde{u}_5$

$$\tilde{u}_5 = c\left(m\right)\chi_{R_0}\left|x\right|^{5-2m}\left(\sum_{h=1}^{K}\phi_{2h+1}\right)$$

and we get

$$\frac{1}{2}\Delta^2_\eta\left(\chi_{R_0}p_5-\tilde{u}_5\right)=\mathcal{O}\left(|x|^{-1-2m}\right),$$

and it is then easy to see that

$$\frac{1}{2}\int_X\Delta^2_\eta\left(\chi_{R_0}p_5-\tilde{u}_5\right)\,d\mu_\eta=0\,.$$

Again by Propositions 2.6 and 2.7 we can find a function $\bar{u}_5\in C^{4,\alpha}_{3-2m+\delta'}\left(X\right)$ such that

$$\frac{1}{2}\Delta^2_\eta\bar{u}_5=\frac{1}{2}\Delta^2_\eta\left(\chi_{R_0}p_5-\tilde{u}_5\right),$$

we define moreover

$$u_5:=\tilde{u}_5+\bar{u}_5\,.$$

We now define our function $\mathbb{J}_{\hat{b}}$

$$\begin{aligned}\mathbb{J}_{\hat{b}}:=&\hat{b}^4\varepsilon^2\left(\chi_{R_0}p_4\left(x\right)-u_4+\frac{c_\Gamma\left(m-1\right)s_g}{2\left(m-2\right)m\left(m+1\right)}\chi_{R_0}|x|^{4-2m}\right)\\ &+\hat{b}^5\varepsilon^3\left(\chi_{R_0}p_5\left(x\right)-u_5\right)+\frac{\tilde{\chi}_{R_0}}{\varepsilon^2}\left(\sum_{k=2}^{+\infty}p_{4+k}\left(\hat{b}\varepsilon x\right)\right).\end{aligned}\tag{3.3}$$

This choice for the skeleton solution is “natural” since we are perturbing the background metric on a big compact set “near infinity” with the whole potential (suitably rescaled) of a *CscK*-metric. The resulting metric is, indeed, “near” to a *Csck*-metric. This construction is essentially different from the one in [AP09] and it is inspired to the construction Szekelyhidi performs in [Szé12].

3.2.2 Construction of $\tilde{H}^I_{\tilde{h}\tilde{k}}$

This is step 14. In this setting we take $\left(\tilde{h},\tilde{k}\right)\in C^{4,\alpha}(\partial B_1)\times C^{2,\alpha}(\partial B_1)$ that are Γ-invariant and such that their means $\tilde{h}^{(0)},\tilde{k}^{(0)}$

$$\tilde{h}^{(0)}:=\frac{1}{\mu\left(S^{2m-1}\right)}\int_{S^{2m-1}}\tilde{h}d\mu_0\qquad\tilde{k}^{(0)}:=\frac{1}{\mu\left(S^{2m-1}\right)}\int_{S^{2m-1}}\tilde{k}d\mu_0$$

satisfy the estimate

$$\left|\tilde{h}^{(0)}\right|, \left|\tilde{k}^{(0)}\right| \leq \kappa r_\varepsilon^\beta \varepsilon^{-2}$$

and their "non-radial parts" $\tilde{h}^{(\dagger)}, \tilde{k}^{(\dagger)}$

$$\tilde{h}^{(\dagger)} := \tilde{h} - \tilde{h}^{(0)} \qquad \tilde{k}^{(\dagger)} := \tilde{k} - \tilde{k}^{(0)}$$

satisfy the estimate

$$\left\|\tilde{h}^{(\dagger)}\right\|_{C^{4,\alpha}(S^{2m-1})}, \left\|\tilde{k}^{(\dagger)}\right\|_{C^{2,\alpha}(S^{2m-1})} \leq \kappa r_\varepsilon^\sigma \varepsilon^{-2}$$

with

$$\begin{aligned} r_\varepsilon^\sigma &= \varepsilon^{2m+4} r_\varepsilon^{-2-2m-\tau} \qquad \tau > 0 \\ r_\varepsilon^\beta &= \varepsilon^{4m+2} r_\varepsilon^{-4m-\tau} \qquad \tau > 0 \end{aligned}$$

and $\kappa \in \mathbb{R}^+$ to be determined. We set

$$\begin{aligned} R_\varepsilon^{-\beta'} &= r_\varepsilon^\beta \varepsilon^{-2} \\ R_\varepsilon^{-\sigma'} &= r_\varepsilon^\sigma \varepsilon^{-2} \end{aligned}$$

and by abuse of notation, we set

$$\mathcal{B}_\alpha := \left\{ \left(\tilde{h}, \tilde{k}\right) \in C^{4,\alpha}(\partial B_1) \times C^{2,\alpha}(\partial B_1) \,|\, \tilde{h}, \tilde{k} \text{ are } \Gamma - \text{invariant} \right\}.$$

If $\left(\tilde{h}, \tilde{k}\right) \in \mathcal{B}_\alpha$ satisfy conditions above we will write

$$\left(\tilde{h}, \tilde{k}\right) \in \mathfrak{B}\left(\kappa, \beta', \sigma'\right)$$

We consider the biharmonic extension $H^I_{\tilde{h}\tilde{k}}$ on B_1 of $\tilde{h}, \tilde{k} \in \mathcal{B}_\alpha$ given by the solution of the boundary value problem

$$\begin{cases} \Delta^2 H^I_{\tilde{h}\tilde{k}} = 0 & w \in B_1 \\ H^I_{\tilde{h}\tilde{k}} = \tilde{h} & w \in \partial B_1 \\ \Delta H^I_{\tilde{h}\tilde{k}} = \tilde{k} & w \in \partial B_1 \end{cases}$$

that has the following expansion

$$H^I_{\tilde{h}\tilde{k}}(w) = \sum_{\gamma=0}^{+\infty} \left(\left(\tilde{h}^{(\gamma)} - \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)} \right) |w|^\gamma + \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)} |w|^{\gamma+2} \right) \phi_\gamma .$$

We recall that with $\tilde{h}^{(\gamma)}\phi_\gamma, \tilde{k}^{(\gamma)}\phi_\gamma$ we mean the projection of h, k onto the γ-th eigenspace of $\Delta_{S^{2m-1}}$ with the orthonormal basis $\left\{\bar{\phi}_{\gamma,1}, \ldots, \bar{\phi}_{\gamma,N_\gamma}\right\}$. We recall also that if the group Γ is non trivial then there is no ϕ_1 in the above summations and so we have

$$H^I_{h,k} = \left(\tilde{h}^{(0)} - \frac{\tilde{k}^{(0)}}{4m} \right) + \frac{\tilde{k}^{(0)}}{4m}|w|^2 + \sum_{\gamma=2}^{+\infty} \left(\left(\tilde{h}^{(\gamma)} - \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)} \right) |w|^\gamma + \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)} |w|^{\gamma+2} \right) \phi_\gamma$$

We want to build a function on X that is "almost" in the kernel of $\mathbb{L}_\eta$ and, for our purposes, we need a more refined construction than [AP06],[AP09]. We consider on $X_{\frac{R_0}{2}}$ the function

$$\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}|x|^2\right) .$$

We have that

$$\begin{aligned}\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}|x|^2\right) =&\mathbb{L}_\eta\left(\chi_{R_0}|x|^2\right)\\ =&\left(\frac{\Delta^2}{2}+\tilde{\mathbb{L}}_\eta\right)\left(\chi_{R_0}|x|^2\right)\\ =&\tilde{\mathbb{L}}_\eta\left(\chi_{R_0}|x|^2\right)\\ =&\mathcal{O}\left(|x|^{-2-2m}\right) ,\end{aligned}$$

so $\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}|x|^2\right)\in C^{0,\alpha}_{-2m-2}(X)$ and therefore it belongs to $C^{0,\alpha}_{-2m-2+\delta'}(X)$ with $\delta'\in(0,1)$. We want to solve for $u_2^0\in C^{4,\alpha}_{2-2m+\delta'}(X)$ the equation

$$\frac{1}{2}\Delta_\eta^2 u_2^0=\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}|x|^2\right) .$$

By Proposition 2.6 we have to check if the right hand side of the equation has vanishing integral. We have

$$\begin{aligned}\Delta_\eta|x|^2 =&\Delta|x|^2+8c_\Gamma\partial_j\overline{\partial}_i|x|^{2-2m}\partial_i\overline{\partial}_j|x|^2+\mathcal{O}\left(|x|^{-2-2m}\right)\\ =&4m+8c_\Gamma\partial_j\overline{\partial}_i|x|^{2-2m}\delta_{i\bar{j}}+\mathcal{O}\left(|x|^{-2-2m}\right)\\ =&4m+2c_\Gamma\Delta|x|^{2-2m}+\mathcal{O}\left(|x|^{-2-2m}\right)\\ =&4m+\mathcal{O}\left(|x|^{-2-2m}\right) ,\end{aligned}$$

now it is immediate to see that

$$\frac{1}{2}\int_X\Delta_\eta^2\left(\chi_{R_0}|x|^2\right)d\mu_\eta=\lim_{\rho\to+\infty}\frac{1}{2}\int_{X_\rho}\partial_\nu\Delta_\eta\left(\chi_{R_0}|x|^2\right)d\mu_\eta=0$$

so by Theorem 2.6 exists $u_2^0\in C^{4,\alpha}_{2-2m+\delta'}(X)$. We want to do the same thing for functions

$$\chi_{R_0}|x|^2\phi_2$$

with ϕ_2 eigenfunction of eigenvalue $-4m$ of the euclidean Laplace operator and

$$\chi_{R_0}|x|^3\phi_3$$

with ϕ_3 eigenfunction of eigenvalue $3(2m+1)$ of the euclidean Laplace operator. We have

$$\begin{aligned}\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}|x|^2\phi_2\right)=&\left(\frac{\Delta^2}{2}+\tilde{\mathbb{L}}_\eta\right)\left(|x|^2\phi_2\right)\\ =&\tilde{\mathbb{L}}_\eta\left(|x|^2\phi_2\right)\\ =&\mathcal{O}\left(|x|^{-2-2m}\right)\end{aligned}$$

and since $|x|^2\phi_2 = P_{i\bar{j}}x^i\overline{x^j}$ or $|x|^2\phi_2 = P_{ij}x^ix^j + \overline{P_{ij}x^ix^j}$ with $P_{i\bar{j}}, P_{ij} \in \mathbb{C}$

$$\begin{aligned}\Delta_\eta|x|^2\phi_2 =&\Delta\left(|x|^2\phi_2\right) + 8c_\Gamma\partial_j\overline{\partial}_i|x|^{2-2m}\partial_i\overline{\partial}_j\left(|x|^2\phi_2\right) + \mathcal{O}\left(|x|^{-2-2m}\right)\\ =&8c_\Gamma\partial_j\overline{\partial}_i|x|^{2-2m}P_{i\bar{j}} + \mathcal{O}\left(|x|^{-2-2m}\right)\\ =&\mathcal{O}\left(|x|^{-2m}\right)\,.\end{aligned}$$

It is immediate to see that

$$\frac{1}{2}\int_X \Delta_\eta^2\left(\chi_{R_0}|x|^2\phi_2\right)d\mu_\eta = \lim_{\rho\to+\infty}\frac{1}{2}\int_{X_\rho}\partial_\nu\Delta_\eta\left(\chi_{R_0}|x|^2\phi_2\right)d\mu_\eta = 0$$

so by Theorem 2.6 exists $u_2^2 \in C^{4,\alpha}_{2-2m+\delta'}(X)$. Now we deal with the last type of function and we note that $\rho^3\phi_3 = P_3(x,\overline{x})$ with P_3 a harmonic homogeneous real polynomial of degree 3.

$$\begin{aligned}\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}|x|^3\phi_3\right) &= \left(\frac{\Delta^2}{2} + \tilde{\mathbb{L}}_\eta\right)\left(\chi_{R_0}|x|^3\phi_3\right)\\ &= \tilde{\mathbb{L}}_\eta\left(\chi_{R_0}|x|^3\phi_3\right)\\ &= \mathcal{O}\left(|x|^{-1-2m}\left(\sum_{h=1}^{K}\phi_{2h+1}\right)\right)\,.\end{aligned}$$

We correct "by hand" $\chi_{R_0}|x|^3\phi_3$ subtracting a term $\tilde{u}_3$

$$\tilde{u}_3^3 = c(m)\,\chi_{R_0}\,|x|^{3-2m}\left(\sum_{h=1}^{K}\phi_{2h+1}\right)$$

and we get

$$\frac{1}{2}\Delta_\eta^2\left(\chi_{R_0}|x|^3\phi_3 - \tilde{u}_3^3\right) = \mathcal{O}\left(|x|^{-3-2m}\right)\,,$$

so to apply Proposition 2.6 we have to compute the quantity

$$\frac{1}{2}\int_X \Delta_\eta^2\left(\chi_{R_0}|x|^3\phi_3 - \tilde{u}_3^3\right)d\mu_\eta\,.$$

We have

$$\begin{aligned}\Delta_\eta\left(|x|^3\phi_3 - \tilde{u}_3^3\right) =&\Delta\left(|x|^3\phi_3 - \tilde{u}_3^3\right) + 8c_\Gamma\partial_j\overline{\partial}_i|x|^{2-2m}\partial_i\overline{\partial}_j\left(|x|^3\phi_3 - \tilde{u}_3^3\right) + \mathcal{O}\left(|x|^{-1-2m}\right)\\ =&-\Delta\tilde{u}_3^3 + 8c_\Gamma\partial_j\overline{\partial}_i|x|^{2-2m}\partial_i\overline{\partial}_j\left(P_3 - \tilde{u}_3^3\right) + \mathcal{O}\left(|x|^{-1-2m}\right)\\ =&\mathcal{O}\left(|x|^{1-2m}\right)\sum_{l=1}^{L}\phi_{2l+1} + \mathcal{O}\left(|x|^{-1-2m}\right)\end{aligned}$$

and it is immediate to see that

$$\frac{1}{2}\int_X \Delta_\eta^2\left(\chi_{R_0}|x|^3\phi_3 - \tilde{u}_3^3\right)d\mu_\eta = \lim_{\rho\to+\infty}\frac{1}{2}\int_{X_\rho}\partial_\nu\Delta_\eta\left(\chi_{R_0}|x|^3\phi_3 - \tilde{u}_3^3\right)d\mu_\eta = 0$$

so by Theorem 2.6 exists $\bar{u}_3^3 \in C^{4,\alpha}_{2-2m+\delta'}(X)$ such that

$$\frac{1}{2}\Delta_\eta^2 \bar{u}_3^3 = -\frac{1}{2}\Delta_\eta^2 \left(\chi_{R_0}|x|^3\phi_3 - \tilde{u}_3^3\right)$$

and we define

$$u_3^3 := \bar{u}_3^3 + \tilde{u}_3^3 .$$

We are ready to define the function $\tilde{H}^I_{\tilde{h}\tilde{k}} \in C^{4,\alpha}(X_{\frac{R_\varepsilon}{\hat{b}}})$

$$\begin{aligned}
\tilde{H}^I_{\tilde{h}\tilde{k}} :=& \left(\tilde{h}^{(0)} - \frac{\tilde{k}^{(0)}}{4m}\right) + \frac{\tilde{k}^{(0)}\hat{b}^2}{4mR_\varepsilon^2}\left(\chi_{R_0}|x|^2 - u_2^0\right) \\
&+ \left[\left(\tilde{h}^{(2)} - \frac{\tilde{k}^{(2)}}{4(m+2)}\right)\frac{\hat{b}^2\chi_{R_0}|x|^2}{R_\varepsilon^2} + \frac{\tilde{k}^{(2)}\hat{b}^4}{4(m+2)R_\varepsilon^4}\chi_{R_0}|x|^4\right]\phi_2 - \left(\tilde{h}^{(2)} - \frac{\tilde{k}^{(2)}}{4(m+2)}\right)\frac{\hat{b}^2u_2^2}{R_\varepsilon^2} \\
&+ \left[\left(\tilde{h}^{(3)} - \frac{\tilde{k}^{(3)}}{4(m+3)}\right)\frac{\hat{b}^3\chi_{R_0}|x|^3}{R_\varepsilon^3} + \frac{\tilde{k}^{(3)}\hat{b}^5}{4(m+3)R_\varepsilon^5}\chi_{R_0}|x|^5\right]\phi_3 - \left(\tilde{h}^{(3)} - \frac{\tilde{k}^{(3)}\hat{b}^3}{4(m+3)}\right)\frac{u_3^3}{R_\varepsilon^3} \\
&+ \chi_{R_0}\left(\sum_{\gamma=4}^{+\infty}\left(\left(\tilde{h}^{(\gamma)} - \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)}\right)\left|\frac{\hat{b}x}{R_\varepsilon}\right|^\gamma + \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)}\left|\frac{\hat{b}x}{R_\varepsilon}\right|^{\gamma+2}\right)\phi_\gamma\right) .
\end{aligned} \tag{3.4}$$

3.2.3 Construction of $f^I_{\tilde{h}\tilde{k}}$

We have come to step 15. Now that we have $\mathbb{J}_{\hat{b}}$ and $\tilde{H}^I_{\tilde{h}\tilde{k}}$, we want to find the last building block: $f^I_{\tilde{h}\tilde{k}}$. We want to find an equation for $f^I_{\tilde{h}\tilde{k}}$ since it has to assure the constancy of scalar curvature on $X_{\frac{R_\varepsilon}{\hat{b}}}$. We want to solve the following problem on $X_{\frac{R_\varepsilon}{\hat{b}}}$

$$s_{\frac{\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}}{\varepsilon^2}} = \varepsilon^2(s_g + \nu) .$$

Since we want to construct $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ as a small perturbation of $\varepsilon^2\eta_j$ we can expand the first term and we have

$$s_\eta - \frac{1}{\hat{b}^4}\mathbb{L}_\eta\tilde{H}^I_{\tilde{h}\tilde{k}} - \frac{1}{\hat{b}^4}\mathbb{L}_\eta f^I_{\tilde{h}\tilde{k}} - \frac{1}{\hat{b}^4}\mathbb{L}_\eta\mathbb{J}_{\hat{b}} + Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}} + \tilde{H}^I_{\tilde{h}\tilde{k}} + f^I_{\tilde{h}\tilde{k}}\right) = \varepsilon^2(s_g + \nu)$$

that after simplifying and reordering becomes

$$\mathbb{L}_\eta f^I_{\tilde{h}\tilde{k}} = -\varepsilon^2\hat{b}^4(s_g + \nu) - \mathbb{L}_\eta\tilde{H}^I_{\tilde{h}\tilde{k}} - \mathbb{L}_\eta\mathbb{J}_{\hat{b}} + \hat{b}^4Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}} + \tilde{H}^I_{\tilde{h}\tilde{k}} + f^I_{\tilde{h}\tilde{k}}\right) .$$

We have to solve the equation above in $X_{\frac{R_\varepsilon}{\hat{b}}}$ and as in the case of the base manifold we modify slightly the problem to solve an equation on weighte Hölder spaces on the whole X. To this aim we introduce an operator truncation/extension between weighted spaces.

Definition 3.4. Let $f \in C^{0,\alpha}_\delta(X)$, we define $\mathcal{E}_R : C^{0,\alpha}_\delta(X) \to C^{0,\alpha}_\delta(X)$

$$\mathcal{E}_R(f) : \begin{cases} f(x) & x \in X_R \\ f\left(R\frac{x}{|x|}\right)\chi_2\left(\frac{|x|}{R}\right) & x \in X_{2R} \setminus X_R \\ 0 & x \in X \setminus X_{2R} \end{cases}$$

The equation for $f^I_{\tilde{h}\tilde{k}}$ becomes

$$\mathbb{L}_\eta f^I_{\tilde{h}\tilde{k}} = -\varepsilon^2 \hat{b}^4 \mathcal{E}_{R_\varepsilon}(s_g+\nu) - \mathcal{E}_{R_\varepsilon}\mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}} - \mathcal{E}_{R_\varepsilon}\mathbb{L}_\eta \mathbb{J}_{\hat{b}} + \hat{b}^4 \mathcal{E}_{R_\varepsilon} Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}} + \tilde{H}^I_{\tilde{h}\tilde{k}} + f^I_{\tilde{h}\tilde{k}}\right) .$$

Now let

$$\mathbb{G}_\delta : C^{0,\alpha}_{-2m+\delta}(X) \to C^{4,\alpha}_{4-2m+\delta}(X)$$

be the inverse of $\mathbb{L}_\eta$ whose existence is guaranteed by Proposition 2.6, we define a fixed point problem

$$\begin{aligned} f^I_{\tilde{h}\tilde{k}} =& -\varepsilon^2 \hat{b}^4 \mathbb{G}_\delta \mathcal{E}_{R_\varepsilon}(s_g+\nu) - \mathbb{G}_\delta \mathcal{E}_{R_\varepsilon}\mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}} - \mathbb{G}_\delta \mathcal{E}_{R_\varepsilon}\mathbb{L}_\eta \mathbb{J}_{\hat{b}} + \hat{b}^2 \mathbb{G}_\delta \mathcal{E}_{R_\varepsilon} Q_\eta\left(\mathbb{J}_{\hat{b}} + \tilde{H}^I_{\tilde{h}\tilde{k}} + f^I_{\tilde{h}\tilde{k}}\right) \\ =& \mathcal{N}\left(\varepsilon, \tilde{h}, \tilde{k}, f^I_{\tilde{h}\tilde{k}}\right) . \end{aligned}$$

To get more refined estimates we consider the following equivalent form of the operator $\mathcal{N}$

$$\begin{aligned} \mathcal{N}(\varepsilon, h, k, f^I_{\tilde{h}\tilde{k}}) =& -\varepsilon^2 \hat{b}^4 \mathbb{G}_\delta \mathcal{E}_{R_\varepsilon}\nu - \mathbb{G}_\delta \mathcal{E}_{R_\varepsilon}\mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}} - \mathbb{G}_\delta \mathcal{E}_{R_\varepsilon}\left(\varepsilon^2 s_g + \mathbb{L}_\eta \mathbb{J}_{\hat{b}} - \hat{b}^4 Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right)\right) \\ &+ \hat{b}^4 \mathbb{G}_\delta \mathcal{E}_{R_\varepsilon}\left(Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}} + \tilde{H}^I_{\tilde{h}\tilde{k}} + f^I_{\tilde{h}\tilde{k}}\right) - Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right)\right) . \end{aligned}$$

From now on, we will take δ

$$\delta \in \left(0, \frac{1}{(m+2)^2}\right) .$$

We make this choice of δ in order to get, when we will perform the data matching, estimates (4.5) and (4.6). We are now in position to prove that the operator $\mathcal{N}$ is a contraction on a subset of $C^{4,\alpha}_{4+\delta-2m}(X_j)$ to use the Banach fixed point theorem.

Lemma 3.8. *Let* $\left(\tilde{h}, \tilde{k}\right) \in \mathfrak{B}(\kappa, \beta', \sigma')$, *then the following estimate holds*

$$\left\|\mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}}\right\|_{C^{0,\alpha}_{\delta-2m}(X)} \le C(\eta) \frac{\left\|\tilde{h}^{(\dagger)}, \tilde{k}^{(\dagger)}\right\|_{\mathcal{B}_\alpha}}{R_\varepsilon^4} .$$

Proof. On $X_{\frac{R_0}{\hat{b}}}$ using formula (3.4) we have

$$\begin{aligned} \mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}} =& \frac{\tilde{k}^{(0)}\hat{b}^2}{4mR_\varepsilon^2}\mathbb{L}_\eta\left(\chi_{R_0}|x|^2 - u_2^0\right) \\ &+ \frac{\hat{b}^2}{R_\varepsilon^2}\left(\tilde{h}^{(2)} - \frac{\tilde{k}^{(2)}}{4(m+2)}\right)\mathbb{L}_\eta\left(\chi_{R_0}|x|^2\phi_2 - u_2^2\right) + \frac{\tilde{k}^{(2)}\hat{b}^4}{4(m+2)R_\varepsilon^4}\mathbb{L}_\eta\left(\chi_{R_0}|x|^4\phi_2\right) \\ &+ \frac{\hat{b}^3}{R_\varepsilon^3}\left(\tilde{h}^{(3)} - \frac{\tilde{k}^{(3)}}{4(m+3)}\right)\mathbb{L}_\eta\left(\chi_{R_0}|x|^3\phi_3 - u_3^3\right) + \frac{\tilde{k}^{(3)}\hat{b}^5}{4(m+3)R_\varepsilon^5}\mathbb{L}_\eta\left(\chi_{R_0}|x|^5\phi_3\right) \\ &+ \mathbb{L}_\eta\left(\chi_{R_0}\left(\sum_{\gamma=4}^{+\infty}\left(\left(\tilde{h}^{(\gamma)} - \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)}\right)\left|\frac{\hat{b}x}{R_\varepsilon}\right|^\gamma + \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)}\left|\frac{\hat{b}x}{R_\varepsilon}\right|^{\gamma+2}\right)\phi_\gamma\right)\right) \\ =& \frac{\tilde{k}^{(2)}\hat{b}^4}{4(m+2)R_\varepsilon^4}\mathbb{L}_\eta\left(\chi_{R_0}|x|^4\phi_2\right) + \frac{\tilde{k}^{(3)}\hat{b}^5}{4(m+3)R_\varepsilon^5}\mathbb{L}_\eta\left(\chi_{R_0}|x|^5\phi_3\right) \\ &+ \mathbb{L}_\eta\left(\chi_{R_0}\left(\sum_{\gamma=4}^{+\infty}\left[\left(\tilde{h}^{(\gamma)} - \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)}\right)\left|\frac{\hat{b}x}{R_\varepsilon}\right|^\gamma + \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)}\left|\frac{\hat{b}x}{R_\varepsilon}\right|^{\gamma+2}\right]\phi_\gamma\right)\right) \end{aligned}$$

and so we evince that

$$\left\|\mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}}\right\|_{C^{0,\alpha}\left(X_{\frac{R_0}{\hat{b}}}\right)} \leq C\left(\eta\right) \frac{\left\|\tilde{h}^{(\dagger)}, \tilde{k}^{(\dagger)}\right\|_{\mathcal{B}_\alpha}}{R_\varepsilon^4}\,.$$

Now we estimate the quantity

$$\sup_{\rho\in[R_0,R_\varepsilon]} \rho^{2m-\delta}\left\|\mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}}\left(\rho\cdot\right)\right\|_{C^{0,\alpha}\left(A_1^\Gamma\right)}\,.$$

On $X_{\frac{\rho}{\hat{b}}} \setminus X_{\frac{\rho}{2\hat{b}}}$, using again formula (3.4), we have

$$\begin{aligned}
\mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}}\left(\rho w\right) =& \frac{\tilde{k}^{(2)}\hat{b}^4\rho^4}{4(m+2)R_\varepsilon^4}\tilde{\mathbb{L}}_\eta\left(|w|^4\phi_2\right) + \frac{\tilde{k}^{(3)}\hat{b}^5\rho^5}{4(m+3)R_\varepsilon^5}\tilde{\mathbb{L}}_\eta\left(|w|^5\phi_3\right)\\
&+\tilde{\mathbb{L}}_\eta\left(\sum_{\gamma=4}^{+\infty}\left[\left(\tilde{h}^{(\gamma)} - \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)}\right)\left|\frac{\hat{b}\rho w}{R_\varepsilon}\right|^\gamma + \frac{\tilde{k}^{(\gamma)}}{4(m+\gamma)}\left|\frac{\hat{b}\rho w}{R_\varepsilon}\right|^{\gamma+2}\right]\phi_\gamma\right)\\
=&\frac{\left\|\tilde{h}^{(\dagger)}, \tilde{k}^{(\dagger)}\right\|}{R_\varepsilon^4}\rho^{-2m}\mathcal{O}_{C^{0,\alpha}\left(A_1^\Gamma\right)}\left(1+\frac{\rho}{R_\varepsilon}\right)\,.
\end{aligned}$$

So we have

$$\sup_{\rho\in[R_0,R_\varepsilon]} \rho^{2m-\delta}\left\|\mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}}\left(\rho w\right)\right\|_{C^{0,\alpha}\left(A_1^\Gamma\right)} \leq C\left(\eta\right)\frac{\left\|\tilde{h}^{(\dagger)}, \tilde{k}^{(\dagger)}\right\|_{\mathcal{B}_\alpha}}{R_\varepsilon^{4+\delta}}\,.$$

□

Lemma 3.9. *The following estimate holds*

$$\left\|\mathcal{E}_{R_\varepsilon}\left(-\varepsilon^2\hat{b}^4 s_g - \mathbb{L}_\eta \mathbb{J}_{\hat{b}} + \hat{b}^4 Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right)\right)\right\|_{C^{0,\alpha}_{\delta-2m}(X)} \leq C\left(g,\eta\right)\varepsilon^4 R_\varepsilon^{2-\delta}$$

Proof. We recall the structure of $\mathbb{J}_{\hat{b}_j}$ given in formula (3.1). On $X_{\frac{R_0}{\hat{b}}}$ we have

$$-\varepsilon^2\hat{b}^4 s_g - \mathbb{L}_\eta\mathbb{J}_{\hat{b}} + \hat{b}^4 Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right) = -\varepsilon^2\hat{b}^4 s_g - \varepsilon^2\hat{b}^4\mathbb{L}_\eta\left(\chi_{R_0}p_4\left(x\right) - u_4\right) - \varepsilon^3\hat{b}^5\mathbb{L}_\eta\left(\chi_{R_0}p_5\left(x\right) - u_5\right)$$

$$\begin{aligned}
&+\mathbb{L}_\eta\left(\frac{\tilde{\chi}_{R_0}}{\varepsilon^2}\left(\sum_{k=2}^{+\infty}p_{4+k}\left(\hat{b}\varepsilon x\right)\right)\right) + \hat{b}^4 Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right)\\
=&\mathbb{L}_\eta\left(\frac{\tilde{\chi}_{R_0}}{\varepsilon^2}\left(\sum_{k=2}^{+\infty}p_{4+k}\left(\hat{b}\varepsilon z\right)\right)\right) + \hat{b}^4 Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right)\\
=&\varepsilon^4\mathcal{O}_{C^{0,\alpha}\left(X_{\frac{R_0}{\hat{b}}}\right)}(1)
\end{aligned}$$

and so

$$\left\|-\varepsilon^2\hat{b}^4 s_g - \mathbb{L}_\eta\mathbb{J}_{\hat{b}} + \hat{b}^4 Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right)\right\|_{C^{0,\alpha}\left(X_{\frac{R_0}{\hat{b}}}\right)} \leq C\left(g,\eta\right)\varepsilon^4\,.$$

Now we estimate the weighted part of the norm. On $X_{\frac{R_\varepsilon}{\hat{b}}} \setminus X_{\frac{R_0}{2\hat{b}}}$, using Proposition 1.3 we have

$$\begin{aligned}
-\varepsilon^2\hat{b}^4 s_g - \mathbb{L}_\eta \mathbb{J}_{\hat{b}} + \hat{b}^4 Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right) =& \hat{b}^2 \mathbf{s}_\eta \left(\frac{\mathbb{J}_{\hat{b}}}{\hat{b}^2}\right) - \varepsilon^2 \hat{b}^4 s_g \\
=& -\varepsilon^2 \hat{b}^4 s_g + \hat{b}^2 \sum_{k=1}^{+\infty} \frac{1}{k!} \mathbf{s}_\eta^k \left(\frac{\mathbb{J}_{\hat{b}}}{\hat{b}^2}\right) \\
=& \hat{b}^2 \sum_{k=1}^{\infty} \frac{1}{k!} \mathbf{s}_\eta^k \left(\frac{\mathbb{J}_{\hat{b}}}{\hat{b}^2}\right) - \hat{b}^2 \mathbf{s}_0 \left(\frac{1}{\varepsilon^2 \hat{b}^2} \psi_g\left(\hat{b}\varepsilon x\right)\right) \\
=& \hat{b}^2 \mathbf{s}_\eta^1 \left(\frac{1}{\varepsilon^2\hat{b}^2}\left(\psi_g\left(\hat{b}\varepsilon x\right) - p_4\left(\hat{b}\varepsilon x\right) - p_5\left(\hat{b}\varepsilon x\right)\right)\right) \\
& - \hat{b}^2 \mathbf{s}_0^1 \left(\frac{1}{\varepsilon^2\hat{b}^2}\left(\psi_g\left(\hat{b}\varepsilon x\right) - p_4\left(\hat{b}\varepsilon x\right) - p_5\left(\hat{b}\varepsilon x\right)\right)\right) \\
& + \hat{b}^2 \sum_{k=2}^{+\infty} \frac{1}{k!} \left[\mathbf{s}_\eta^k \left(\frac{\mathbb{J}_{\hat{b}}}{\hat{b}^2}\right) - \mathbf{s}_0^k \left(\frac{1}{\varepsilon^2\hat{b}^2}\left(\psi_g\left(\hat{b}\varepsilon x\right)\right)\right)\right].
\end{aligned}$$

And so, on $X_{\frac{\rho}{\hat{b}}} \setminus X_{\frac{\rho}{2\hat{b}}}$ we have

$$\left[-\varepsilon^2\hat{b}^4 s_g - \mathbb{L}_\eta \mathbb{J}_{\hat{b}} + \hat{b}^4 Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right)\right]\left(\frac{\rho}{\hat{b}} w\right) = \varepsilon^4 \rho^{2-2m} \mathcal{O}_{C^{0,\alpha}\left(A_1^\Gamma\right)}\left(1+\varepsilon\rho\right)$$

and we can conclude

$$\sup_{\rho\in[R_0,R_\varepsilon]} \rho^{2m-\delta} \left\| \left[-\varepsilon^2\hat{b}^4 s_g - \mathbb{L}_\eta \mathbb{J}_{\hat{b}} + \hat{b}^4 Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right)\right]\left(\frac{\rho}{\hat{b}} w\right)\right\|_{C^{0,\alpha}\left(A_1^\Gamma\right)} \leq C\left(g,\eta\right)\varepsilon^4 R_\varepsilon^{2-\delta}.$$

□

We now fix

$$C_f R_\varepsilon^{-\mu'} = 2\left\|\mathbb{L}_\eta \tilde{H}^I_{\tilde{h}\tilde{k}}\right\|_{C^{0,\alpha}_{\delta-2m}(X)}$$

and so

$$R_\varepsilon^{-\mu'} = R_\varepsilon^{-\sigma'-4}.$$

Lemma 3.10. *Let* $\left(\tilde{h},\tilde{k}\right) \in \mathfrak{B}\left(\kappa,\beta',\sigma'\right)$, $f \in C^{4,\alpha}_{4+\delta-2m}\left(X\right)$ *such that*

$$\left\|f^I_{\tilde{h}\tilde{k}}\right\|_{C^{0,\alpha}_{\delta-2m}(X)} \leq C_f R_\varepsilon^{\mu'}$$

then the following estimate holds

$$\left\|\mathcal{E}_{R_\varepsilon}\left(Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}} + \tilde{H}^I_{\tilde{h}\tilde{k}} + f^I_{\tilde{h}\tilde{k}}\right) - Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right)\right)\right\|_{C^{0,\alpha}_{\delta-2m}(X)} \leq C\left(g,\eta\right)\kappa\varepsilon^2 R_\varepsilon^{2m-2-\delta-\sigma'}$$

Proof. On $X_{\frac{R_0}{\hat{b}}}$, using Lemma 1.3 we have

$$Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}} + \tilde{H}^I_{\tilde{h}\tilde{k}} + f^I_{\tilde{h}\tilde{k}}\right) - Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}\right) = \sum_{k=2}^{+\infty} \frac{1}{k!\hat{b}^{2(k+1)}}\left[\mathbf{s}_\eta^k\left(\mathbb{J}_{\hat{b}} + \tilde{H}^I_{\tilde{h}\tilde{k}} + f^I_{\tilde{h}\tilde{k}}\right) - \mathbf{s}_\eta^k\left(\mathbb{J}_{\hat{b}}\right)\right]$$

and hence suffices to work out estimates of terms

$$\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)-\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}\right).$$

We have

$$\begin{aligned}\left\|\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)-\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}\right)\right\|_{C^{0,\alpha}\left(X_{\frac{R_0}{\hat{b}}}\right)}&=\left(\kappa\varepsilon^{2}R_{\varepsilon}^{-\sigma'-2}+C_{f}\varepsilon^{2}R_{\varepsilon}^{-\mu'}\right)\mathcal{O}_{C^{0,\alpha}\left(A_{1}^{\Gamma}\right)}(1)\\&\quad+\left(\kappa^{2}R_{\varepsilon}^{-2\sigma'-4}+C_{f}^{2}R_{\varepsilon}^{-2\mu'}\right)\mathcal{O}_{C^{0,\alpha}\left(A_{1}^{\Gamma}\right)}(1)\end{aligned}$$

and so on $X_{\frac{R_0}{\hat{b}}}$ we get

$$\begin{aligned}\left\|Q_{\hat{b}^{2}\eta}\left(\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)-Q_{\hat{b}^{2}\eta}\left(\mathbb{J}_{\hat{b}}\right)\right\|_{C^{0,\alpha}\left(X_{\frac{R_0}{\hat{b}}}\right)}&\leq C\left(\eta,g\right)\left(\varepsilon^{2}R_{\varepsilon}^{-\sigma'-2}+\varepsilon^{2}R_{\varepsilon}^{-\mu'}\right)\\&\quad+C\left(\eta,g\right)\left(R_{\varepsilon}^{-2\sigma'-4}+R_{\varepsilon}^{-2\mu'}\right).\end{aligned}$$

We now estimate the weighted part of the norm. On $X_{\rho/\hat{b}}\setminus X_{\rho/(2\hat{b})}$ we have

$$\begin{aligned}\left[Q_{\eta}\left(\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)-Q_{\eta}\left(\mathbb{J}_{\hat{b}}\right)\right]&=\sum_{k=2}^{+\infty}\frac{1}{k!\hat{b}^{2k+2}}\left[\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)-\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}\right)\right]\\&=\sum_{k=2}^{+\infty}\frac{1}{k!\hat{b}^{2k+2}}\left[\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)-\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}\right)\right]\\&=\sum_{k=2}^{+\infty}\frac{1}{k!\hat{b}^{2k+2}}\mathbf{s}_{0}^{k}\left(-c_{\Gamma}|x|^{2-2m}+\psi_{\eta}+\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)\\&\quad-\sum_{k=2}^{+\infty}\frac{1}{k!\hat{b}^{2k+2}}-\mathbf{s}_{0}^{k}\left(-c_{\Gamma}|x|^{2-2m}+\psi_{\eta}+\mathbb{J}_{\hat{b}}\right).\end{aligned}$$

Again we need to estimate on $X_{\frac{\rho}{\hat{b}}}\setminus X_{\frac{\rho}{2\hat{b}}}$ a term of the form

$$\mathbf{s}_{0}^{k}\left(-c_{\Gamma}|x|^{2-2m}+\psi_{\eta}+\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)-\mathbf{s}_{0}^{k}\left(-c_{\Gamma}|x|^{2-2m}+\psi_{\eta}+\mathbb{J}_{\hat{b}}\right).$$

We get

$$\begin{aligned}\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)-\mathbf{s}_{\eta}^{k}\left(\mathbb{J}_{\hat{b}}\right)&=\mathbf{s}_{0}^{k}\left(-c_{\Gamma}|x|^{2-2m}+\psi_{\eta}+\mathbb{J}_{\hat{b}}+\tilde{H}_{\tilde{h}\tilde{k}}^{I}+f_{\tilde{h}\tilde{k}}^{I}\right)\\&\quad-\mathbf{s}_{0}^{k}\left(-c_{\Gamma}|x|^{2-2m}+\psi_{\eta}+\mathbb{J}_{\hat{b}}\right)\\&=\frac{\kappa\varepsilon^{2}R_{\varepsilon}^{-\sigma'}}{\rho^{2}}\mathcal{O}_{C^{0,\alpha}\left(A_{1}^{\Gamma}\right)}\left(1+\frac{\rho}{R_{\varepsilon}}\right)\\&\quad+\frac{C_{f}\varepsilon^{2}R_{\varepsilon}^{-\mu'}}{\rho^{2m-2-\delta}}\mathcal{O}_{C^{0,\alpha}\left(A_{1}^{\Gamma}\right)}\left(1+\frac{\rho}{R_{\varepsilon}}\right)\\&\quad+\frac{C_{f}^{2}R_{\varepsilon}^{-2\mu'}}{\rho^{4m-2-2\delta}}\mathcal{O}_{C^{0,\alpha}\left(A_{1}^{\Gamma}\right)}\left(1+\frac{\rho}{R_{\varepsilon}}\right)\\&\quad+\frac{\kappa^{2}R_{\varepsilon}^{-2\sigma'}}{\rho^{6}}\mathcal{O}_{C^{0,\alpha}\left(A_{1}^{\Gamma}\right)}\left(1+\frac{\rho}{R_{\varepsilon}}\right).\end{aligned}$$

So we have

$$\begin{aligned}\sup_{\rho\in[R_0,R_\varepsilon]}\rho^{2m-\delta}\left\|\left[Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}+\tilde H^I_{\tilde h\tilde k}+f^I_{\tilde h\tilde k}\right)-Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}\right)\right]\right\|_{C^{0,\alpha}\left(A_1^\Gamma\right)} \leq & C\left(\eta,g\right)\varepsilon^2R_\varepsilon^{2m-2-\delta-\sigma'}\\ &+C\left(\eta,g\right)\varepsilon^2R_\varepsilon^{2-\mu'}\\ &+C\left(\eta,g\right)R_\varepsilon^{-2\sigma'+2m-6-\delta}\\ &+C\left(\eta,g\right)R_\varepsilon^{-2\mu'+2-2m+\delta}\,.\end{aligned}$$

□

Now we will study continuity properties of $\mathcal{N}$ with respect to its arguments.

Lemma 3.11. *Let* $\left(\tilde h,\tilde k\right)\in\mathfrak{B}\left(\kappa,\beta',\sigma'\right)$, $f,f'\in C^{4,\alpha}_{4+\delta-2m}\left(X\right)$ *and*

$$\|f\|_{C^{4,\alpha}_{4+\delta-2m}(X)}\,,\|f'\|_{C^{4,\alpha}_{4+\delta-2m}(X)}\leq C_fR_\varepsilon^{-\mu'}\,,$$

then

$$\left\|\mathcal{N}(\varepsilon,\tilde h,\tilde k,f)-\mathcal{N}(\varepsilon,\tilde h,\tilde k,f')\right\|_{C^{4,\alpha}_\delta(X)}\leq\frac{1}{2}\|f-f'\|_{C^{4,\alpha}_{4+\delta-2m}(X)}$$

Proof. We ecall that

$$\mathcal{N}\left(f,\tilde h,\tilde k\right)-\mathcal{N}\left(f',\tilde h,\tilde k\right)=\mathbb{G}_\delta\left[\mathcal{E}_{R_\varepsilon}Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}+\tilde H^I_{\tilde h\tilde k}+f\right)-\mathcal{E}_{R_\varepsilon}Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}+\tilde H^I_{\tilde h\tilde k}+f'\right)\right]\,,$$

so we only need to estimate the quantity

$$\left\|\mathcal{E}_{R_\varepsilon}Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}+\tilde H^I_{\tilde h\tilde k}+f\right)-\mathcal{E}_{R_\varepsilon}Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}+\tilde H^I_{\tilde h\tilde k}+f'\right)\right\|_{C^{0,\alpha}_{\delta-2m}(X)}\,.$$

On $X_{\frac{R_0}{\hat b}}$, using Lemma1.2, we have

$$\begin{aligned}\left\|Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}+\tilde H^I_{\tilde h\tilde k}+f\right)-Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}+\tilde H^I_{\tilde h\tilde k}+f'\right)\right\|_{C^{0,\alpha}\left(X_{\frac{R_0}{\hat b}}\right)}\leq & C\left(\eta\right)\kappa R_\varepsilon^{-\sigma'-2}\|f-f'\|_{C^{4,\alpha}\left(X_{\frac{R_0}{\hat b}}\right)}\\ &+C\left(\eta\right)\epsilon^2\|f-f'\|_{C^{4,\alpha}\left(X_{\frac{R_0}{\hat b}}\right)}\\ &+C\left(\eta\right)C_fR_\varepsilon^{-\mu'}\|f-f'\|_{C^{4,\alpha}\left(X_{\frac{R_0}{\hat b}}\right)}\,.\end{aligned}$$

For the weighted part of the norm we have

$$\sup_{\rho\in[R_0,R_\varepsilon]}\rho^{2m-\tau}\left\|Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}+\tilde H^I_{\tilde h\tilde k}+f\right)-Q_{\hat b^2\eta}\left(\mathbb{J}_{\hat b}+\tilde H^I_{\tilde h\tilde k}+f'\right)\right\|_{C^{0,\alpha}\left(A_1^\Gamma\right)}\leq D\left(\varepsilon\right)\|f-f'\|_{C^{4,\alpha}_{4+\delta-2m}(X)}$$

with

$$D\left(\varepsilon\right)=C\left(g,\eta\right)\left(\varepsilon^2+R_\varepsilon^{-\sigma'-4}+R_\varepsilon^{-\mu'-2m+\delta}\right)\,.$$

By the choice of parameters σ',μ', if we choose ε sufficiently small we have the proposition.

□

Lemma 3.12. *Let* $\left(\tilde{h},\tilde{k}\right)\in\mathfrak{B}\left(\kappa,\beta',\sigma'\right)$, $f\in C^{4,\alpha}_{4+\delta-2m}\left(X\right)$ *and*

$$\|f\|_{C^{4,\alpha}_{4+\delta-2m}(X)}\leq C_f R_\varepsilon^{-\mu'}$$

then

$$\mathcal{N}\left(f,\cdot,\cdot\right):\mathfrak{B}\left(\kappa,\beta',\sigma'\right)\to C^{4,\alpha}_{4+\delta-2m}\left(X\right)$$

is lipschiz.

Proof. By Lemma 3.3 we have

$$\begin{aligned}\left\|\mathcal{E}_{R_\varepsilon}\mathbb{L}_\eta\tilde{H}^I_{\tilde{h}\tilde{k}}-\mathcal{E}_{R_\varepsilon}\mathbb{L}_\eta\tilde{H}^I_{\tilde{h}'\tilde{k}'}\right\|_{C^{0,\alpha}_{\tau-2m}(M_{\mathbf{p}})}&=\left\|\mathcal{E}_{R_\varepsilon}\mathbb{L}_\eta\tilde{H}^I_{\tilde{h}-\tilde{h}',\tilde{k}-\tilde{k}'}\right\|_{C^{0,\alpha}_{\delta-2m}(X)}\\&\leq\frac{C\left(g,\eta\right)}{R_\varepsilon^4}\left\|\tilde{h}-\tilde{h}',\tilde{k}-\tilde{k}'\right\|_{\mathcal{B}_\alpha}\end{aligned}$$

Doing computations completely analogous to Lemma 3.10 we get

$$\left\|\mathcal{E}_{R_\varepsilon}Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}+\tilde{H}^I_{\tilde{h}\tilde{k}}+f\right)-\mathcal{E}_{R_\varepsilon}Q_{\hat{b}^2\eta}\left(\mathbb{J}_{\hat{b}}+\tilde{H}^I_{\tilde{h}'\tilde{k}'}+f\right)\right\|_{C^{0,\alpha}_{\delta-2m}(X)}\leq D\left(\varepsilon\right)\left\|\tilde{h}-\tilde{h}',\tilde{k}-\tilde{k}'\right\|$$

with

$$D\left(\varepsilon\right)=C\left(g,\boldsymbol{\eta}\right)\left(\frac{\varepsilon^2}{R_\varepsilon^2}+R_\varepsilon^{-\mu'-2}+R_\varepsilon^{-\sigma'-4}+\varepsilon^2R_\varepsilon^{2m-\delta-2}+R_\varepsilon^{-\sigma'-4+2m-\delta}\right)$$

and so by the choices of σ',β',μ' the lemma follows.

□

As in the case of the base orbifold, summing up the results we obtained so far, we have the following proposition.

Proposition 3.5. *Let* $\left(\tilde{h}_j,\tilde{k}_j\right)\in\mathfrak{B}\left(\kappa,\beta',\sigma'\right)$ *then there exist* $f^I_{\tilde{h}_j\tilde{k}_j}\in C^{4,\alpha}_{4+\delta-2m}\left(X\right)$ *such that*

$$\left\|f^I_{\tilde{h}_j\tilde{k}_j}\right\|_{C^{4,\alpha}_{4+\delta-2m}(X)}\leq\overline{C}\left(g,\boldsymbol{\eta},\kappa\right)R_\varepsilon^{-\mu'}$$

and on $X_{\frac{R_\varepsilon}{\hat{b}_j},j}$

$$\omega_{\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}}=\varepsilon^2\hat{b}_j^2\omega_{\eta_j}+\partial\overline{\partial}\mathbb{F}^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$$

is a Kähler metric of constant scalar curvature

$$s_{\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}}=\left(s_g+\nu\right).$$

Moreover the metric $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ *depends continuosly on* $\left(\tilde{h}_j,\tilde{k}_j\right)\in\mathfrak{B}\left(\kappa,\beta',\sigma'\right)$.

Chapter 4

Data Matching

We have finally come to step 16. Now that we have the family of metrics $g_{\mathbf{b},\mathbf{hk}}$ on M_{r_ε} and the family of metrics $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ on X_j we proceed with the gluing construction. To do this we'll write

- local potentials $\Psi^o_{\mathbf{b},\mathbf{hk}}$ of $g_{\mathbf{b},\mathbf{hk}}$ on annuli $B_{2r_\varepsilon}(p_j)\setminus B_{r_\varepsilon}(p_j)$, and we'll rescale them in such a way that (passing to the orbifold covering via the covering map $\pi_{\Gamma_j}: B_2 \to B_2/\Gamma_j$) they become functions on the annulus $B_2\setminus B_1$

- local potentials $\Psi^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ of $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ on the annulus $X_{\frac{R_\varepsilon}{\hat{b}_j},j}\setminus B_{\frac{R_\varepsilon}{2\hat{b}_j},j}$ and we'll rescale them in such a way that (pulling back via the covering map $\pi_{\Gamma_j}: B_1 \to B_1/\Gamma_j$) they become functions on the annulus $B_1\setminus B_{\frac{1}{2}}$.

Once we have done this preliminary step the problem will be gluing an N-tuple of functions defined on $B_2\setminus B_1$ with an N-tuple of functions defined on $B_1\setminus B_{\frac{1}{2}}$ to get an N-tuple of "sufficiently regular" functions defined on $B_2\setminus B_{\frac{1}{2}}$. We start renormalizing the potentials.

4.1 Setting up

4.1.1 On X_j

On annuli $X_{\frac{R_\varepsilon}{\hat{b}_j},j}\setminus X_{\frac{R_\varepsilon}{2\hat{b}_j},j}$ we can write the potential $\Psi^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ of $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ in the following way

$$\begin{aligned}\Psi^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}(x) =& \hat{b}_j^2\varepsilon^2\frac{|x|^2}{2} - c_{\Gamma_j}\hat{b}_j^2\varepsilon^2|x|^{2-2m} + \frac{c_{\Gamma_j}\varepsilon^4\hat{b}_j^4(m-1)\,s_g}{2(m-2)\,m\,(m+1)}|x|^{4-2m} + \varepsilon^2\hat{b}_j^2\psi_{\eta_j}(x)\\ &+\psi_g\left(\varepsilon\hat{b}_jx\right)\\ &+\hat{b}_j^4\varepsilon^4|x|^{4-2m}\tilde{\phi}_{2,j}+\hat{b}_j^4\varepsilon^4|x|^{4-2m}\tilde{\phi}_{4,j}+\hat{b}_j^4\varepsilon^4\bar{u}_{4,j}(x)+\varepsilon^5\hat{b}_j^5u_{5,j}(x)\\ &+\varepsilon^2H^I_{\tilde{h}_j\tilde{k}_j}\left(\frac{\hat{b}_jx}{R_\varepsilon}\right)-\frac{\varepsilon^2\hat{b}_j^2\tilde{k}_j^{(0)}}{4mR_\varepsilon^2}u^0_{2,j}(x)-\left(\tilde{h}_j^{(2)}-\frac{\tilde{k}_j^{(2)}}{4(m+2)}\right)\frac{\varepsilon^2\hat{b}_j^2}{R_\varepsilon^2}u^2_{2,j}(x)\\ &-\left(\tilde{h}_j^{(3)}-\frac{\tilde{k}_j^{(3)}}{4(m+3)}\right)\frac{\hat{b}_j^3\varepsilon^2}{R_\varepsilon^3}u^3_{3,j}(x)+f^I_{\tilde{h}_j\tilde{k}_j}(x)\,.\end{aligned}$$

Rescaling $x = \frac{R_\varepsilon}{\hat{b}_j} w$ and recalling that $\varepsilon R_\varepsilon = r_\varepsilon$

$$
\begin{aligned}
\Psi^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) =& \frac{r_\varepsilon^2|w|^2}{2} + \psi_g\left(r_\varepsilon w\right) \\
&- c_{\Gamma_j}\hat{b}_j^{2m}\varepsilon^2 R_\varepsilon^{2-2m}|w|^{2-2m} + \frac{c_{\Gamma_j}\hat{b}_j^{2m}\left(m-1\right)s_g\varepsilon^4 R_\varepsilon^{4-2m}}{2\left(m-2\right)m\left(m+1\right)}|w|^{4-2m} \\
&+ \varepsilon^2\hat{b}_j^2\psi_{\eta_j}\left(\frac{R_\varepsilon w}{\hat{b}_j}\right) \\
&+ \hat{b}_j^4\varepsilon^4 R_\varepsilon^{4-2m}|w|^{4-2m}\tilde{\phi}_{2,j} + \hat{b}_j^4\varepsilon^4 R_\varepsilon^{4-2m}|w|^{4-2m}\tilde{\phi}_{4,j} \\
&+ \hat{b}_j^4\varepsilon^4\bar{u}_{4,j}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) + \varepsilon^5\hat{b}_j^5 u_{5,j}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) \\
&+ \varepsilon^2 H^I_{\tilde{h}_j\tilde{k}_j}\left(w\right) \\
&- \frac{\varepsilon^2\hat{b}_j^2\tilde{k}_j^{(0)}}{4mR_\varepsilon^2}u^0_{2,j}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) - \left(\tilde{h}_j^{(2)} - \frac{\tilde{k}_j^{(2)}}{4(m+2)}\right)\frac{\varepsilon^2\hat{b}_j^2}{R_\varepsilon^2}u^2_{2,j}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) \\
&- \left(\tilde{h}_j^{(3)} - \frac{\tilde{k}_j^{(3)}}{4(m+3)}\right)\frac{\hat{b}_j^3\varepsilon^2}{R_\varepsilon^3}u^3_{3,j}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) + \varepsilon^2 f^I_{\tilde{h}_j\tilde{k}_j}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right).
\end{aligned}
$$

Moreover we make a further refinement of the analysis of the structure of $\Psi^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$: we separate its "radial part" from its "non radial" part.

- The radial part

$$
\begin{aligned}
\left(\Psi^I_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) =& \frac{r_\varepsilon^2|w|^2}{2} + \psi_g^{(0)}\left(r_\varepsilon w\right) \\
&- c_{\Gamma_j}\hat{b}_j^{2m}\varepsilon^2 R_\varepsilon^{2-2m}|w|^{2-2m} + \frac{c_{\Gamma_j}\hat{b}_j^{2m}\left(m-1\right)s_g\varepsilon^4 R_\varepsilon^{4-2m}}{2\left(m-2\right)m\left(m+1\right)}|w|^{4-2m} \\
&+ \varepsilon^2\hat{b}_j^2\psi^{(0)}_{\eta_j}\left(\frac{R_\varepsilon w}{\hat{b}_j}\right) + \hat{b}_j^4\varepsilon^4\bar{u}^{(0)}_{4,j}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) + \varepsilon^5\hat{b}_j^5 u^{(0)}_{5,j}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) \\
&+ \varepsilon^2\left(\tilde{h}_j^{(0)} - \frac{\tilde{k}_j^{(0)}}{4m}\right) + \frac{\varepsilon^2\tilde{k}_j^{(0)}}{4m}|w|^2 \\
&- \frac{\varepsilon^2\hat{b}_j^2\tilde{k}_j^{(0)}}{4mR_\varepsilon^2}\left(u^0_{2,j}\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) - \left(\tilde{h}_j^{(2)} - \frac{\tilde{k}_j^{(2)}}{4(m+2)}\right)\frac{\varepsilon^2\hat{b}_j^2}{R_\varepsilon^2}\left(u^2_{2,j}\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) \\
&- \left(\tilde{h}_j^{(3)} - \frac{\tilde{k}_j^{(3)}}{4(m+3)}\right)\frac{\hat{b}_j^3\varepsilon^2}{R_\varepsilon^3}\left(u^3_{3,j}\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right) + \varepsilon^2\left(f^I_{\tilde{h}_j\tilde{k}_j}\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j}w\right).
\end{aligned}
$$

We can collect some terms and we define

$$
\begin{aligned}
\left(\zeta_j^I\right)^{(0)}(w) :=& \hat{b}_j^4 \varepsilon^4 \bar{u}_{4,j}^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right) + \varepsilon^5 \hat{b}_j^5 u_{5,j}^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right) \\
& - \frac{\varepsilon^2 \hat{b}_j^2 \tilde{k}_j^{(0)}}{4mR_\varepsilon^2}\left(u_{2,j}^0\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right) - \left(\tilde{h}_j^{(2)} - \frac{\tilde{k}_j^{(2)}}{4(m+2)}\right)\frac{\varepsilon^2 \hat{b}_j^2}{R_\varepsilon^2}\left(u_{2,j}^2\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right) \\
& - \left(\tilde{h}_j^{(3)} - \frac{\tilde{k}_j^{(3)}}{4(m+3)}\right)\frac{\hat{b}_j^3 \varepsilon^2}{R_\varepsilon^3}\left(u_{3,j}^3\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right) \\
& + \varepsilon^2\left(f_{\tilde{h}_j \tilde{k}_j}^I\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right)
\end{aligned}
$$

and so we can write

$$
\begin{aligned}
\left(\Psi_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}^I\right)^{(0)}\left(\frac{R_\varepsilon}{\hat{b}_j} w\right) =& \frac{r_\varepsilon^2 |w|^2}{2} + \psi_g^{(0)}\left(r_\varepsilon w\right) \\
& - c_{\Gamma_j} \hat{b}_j^{2m} \varepsilon^2 R_\varepsilon^{2-2m} |w|^{2-2m} + \frac{c_{\Gamma_j} \hat{b}_j^{2m}(m-1) s_g \varepsilon^4 R_\varepsilon^{4-2m}}{2(m-2)m(m+1)} |w|^{4-2m} \\
& + \varepsilon^2 \hat{b}_j^2 \psi_{\eta_j}^{(0)}\left(\frac{R_\varepsilon w}{\hat{b}_j}\right) \\
& + \varepsilon^2\left(\tilde{h}_j^{(0)} - \frac{\tilde{k}_j^{(0)}}{4m}\right) + \frac{\varepsilon^2 \tilde{k}_j^{(0)}}{4m}|w|^2 \\
& + \left(\zeta_j^I\right)^{(0)}(w) \,.
\end{aligned}
$$

The terms written in green will be matched perfectly with their counterpart on M and so we will have to deal with only terms written in black.

- The non radial part

$$
\begin{aligned}
\left(\Psi^I_{\hat b_j,\tilde h_j \tilde k_j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) = & \left(\psi_g\right)^{(\dagger)}\left(r_\varepsilon w\right) \\
& + \varepsilon^2 \hat b_j^2 \left(\psi_{\eta_j}\right)^{(\dagger)}\left(\frac{R_\varepsilon w}{\hat b_j}\right) \\
& + \hat b_j^4 \varepsilon^4 R_\varepsilon^{4-2m}|w|^{4-2m}\tilde\phi_{2,j} + \hat b_j^4 \varepsilon^4 R_\varepsilon^{4-2m}|w|^{4-2m}\tilde\phi_{4,j} \\
& + \varepsilon^4 \hat b_j^4 \left(\bar u_{4,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) - \varepsilon^5 \hat b_j^5 \left(u_{5,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) \\
& + \varepsilon^2 H^I_{\tilde h_j^{(\dagger)}\tilde k_j^{(\dagger)}}(w) - \frac{\varepsilon^2 \hat b_j^2 \tilde k_j^{(0)}}{4mR_\varepsilon^2}\left(u^0_{2,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) \\
& - \left(\tilde h_j^{(2)} - \frac{\tilde k_j^{(2)}}{4(m+2)}\right)\frac{\varepsilon^2 \hat b_j^2}{R_\varepsilon^2}\left(u^2_{2,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) \\
& - \left(\tilde h_j^{(3)} - \frac{\tilde k_j^{(3)}}{4(m+3)}\right)\frac{\hat b_j^3 \varepsilon^2}{R_\varepsilon^3}\left(u^3_{3,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) \\
& + \varepsilon^2\left(f^I_{\tilde h_j \tilde k_j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right).
\end{aligned}
$$

Analogously to the radial part we define

$$
\begin{aligned}
\left(\zeta_j^I\right)^{(\dagger)}(w) := & \hat b_j^4 \varepsilon^4 R_\varepsilon^{4-2m}|w|^{4-2m}\tilde\phi_{2,} + \hat b_j^4 \varepsilon^4 R_\varepsilon^{4-2m}|w|^{4-2m}\tilde\phi_{4,j} \\
& \varepsilon^4 \hat b_j^4 \left(\bar u_{4,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) - \varepsilon^5 \hat b_j^5 \left(u_{5,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) \\
& - \frac{\varepsilon^2 \hat b_j^2 \tilde k_j^{(0)}}{4mR_\varepsilon^2}\left(u^0_{2,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) - \left(\tilde h_j^{(2)} - \frac{\tilde k_j^{(2)}}{4(m+2)}\right)\frac{\varepsilon^2 \hat b_j^2}{R_\varepsilon^2}\left(u^2_{2,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) \\
& - \left(\tilde h_j^{(3)} - \frac{\tilde k_j^{(3)}}{4(m+3)}\right)\frac{\hat b_j^3 \varepsilon^2}{R_\varepsilon^3}\left(u^3_{3,j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) \\
& + \varepsilon^2\left(f^I_{\tilde h_j \tilde k_j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right)
\end{aligned}
$$

and so we can write

$$
\begin{aligned}
\left(\Psi^I_{\hat b_j,\tilde h_j \tilde k_j}\right)^{(\dagger)}\left(\frac{R_\varepsilon}{\hat b_j}w\right) = & \left(\psi_g\right)^{(\dagger)}\left(r_\varepsilon w\right) \\
& + \varepsilon^2 \hat b_j^2 \left(\psi_{\eta_j}\right)^{(\dagger)}\left(\frac{R_\varepsilon w}{\hat b_j}\right) \\
& + \varepsilon^2 H^I_{\tilde h_j^{(\dagger)}\tilde k_j^{(\dagger)}}(w) \\
& + \left(\zeta_j^I\right)^{(\dagger)}(w).
\end{aligned}
$$

The terms written in red will be matched perfectly with their counterpart on M and so we will have to deal with only terms written in black.

4.1.2 On M at points p_j

On annuli $B_{2r_\varepsilon}(p_j) \setminus B_{r_\varepsilon}(p_j)$ we can rewrite $\Psi^o_{\mathbf{b},\mathbf{hk}}$ as

$$
\begin{aligned}
\Psi^o_{\mathbf{b},\mathbf{hk}}(z) =& \frac{|z|^2}{2} + \psi_g(z) \\
&+ c_{\Gamma_j} \bar{b}_j^{2m} \varepsilon^{2m} \left(-|z|^{2-2m} + \frac{(m-1)\, s_g}{2(m-2)\, m\,(m+1)} |z|^{4-2m} \right) \\
&+ \varepsilon^2 \bar{b}_j^2 \psi_{\eta_j}\left(\frac{z}{\varepsilon \bar{b}_j} \right) + \varepsilon^{2m} |z|^{4-2m} \tilde{\phi}_2 \\
&+ \varepsilon^{2m} |z|^{4-2m} \tilde{\phi}_4 + \varepsilon^{2m} |z|^{5-2m} \sum_{h=1}^{K} \tilde{\phi}_{2h+1} + \mathcal{O}\left(\varepsilon^{2m} |z|^{6-2m} \right) \\
&+ H^o_{h_j k_j}\left(\frac{z}{r_\varepsilon} \right) \\
&+ \tilde{f}^o_{\mathbf{b},\mathbf{hk}}(z) \\
&+ c_{\Gamma_j} \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \left(-|z|^{2-2m} + \frac{(m-1)\, s_g}{2(m-2)\, m\,(m+1)} |z|^{4-2m} \right) \\
&+ \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \mathcal{O}\left(|z|^{4-2m} \left(\tilde{\phi}_2 + \tilde{\phi}_4 \right) + |z|^{5-2m} \sum_{h=1}^{K} \tilde{\phi}_{2h+1} + |z|^{6-2m} \right)
\end{aligned}
$$

and rescaling $z = r_\varepsilon w$

$$
\begin{aligned}
\Psi^o_{\mathbf{b},\mathbf{hk}}(r_\varepsilon w) =& \frac{r_\varepsilon^2 |w|^2}{2} + \psi_g(r_\varepsilon w) \\
&+ c_{\Gamma_j} \bar{b}_j^{2m} \varepsilon^{2m} r_\varepsilon^{2-2m} \left(-|w|^{2-2m} + \frac{(m-1)\, s_g r_\varepsilon^2}{2(m-2)\, m\,(m+1)} |w|^{4-2m} \right) \\
&+ \varepsilon^2 \bar{b}_j^2 \psi_{\eta_j}\left(\frac{r_\varepsilon w}{\varepsilon \bar{b}_j} \right) \\
&+ \varepsilon^{2m} r_\varepsilon^{4-2m} |w|^{4-2m} \tilde{\phi}_2 + \varepsilon^{2m} r_\varepsilon^{4-2m} |w|^{4-2m} \tilde{\phi}_4 \\
&+ \varepsilon^{2m} r_\varepsilon^{5-2m} |w|^{5-2m} \sum_{h=1}^{K} \tilde{\phi}_{2h+1} + \mathcal{O}\left(\varepsilon^{2m} r_\varepsilon^{6-2m} |w|^{6-2m} \right) \\
&+ H^o_{h_j k_j}(w) \\
&+ \tilde{f}^o_{\mathbf{b},\mathbf{hk}}(r_\varepsilon w) \\
&+ c_{\Gamma_j} \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} r_\varepsilon^{2-2m} \left(-|w|^{2-2m} + \frac{(m-1)\, s_g r_\varepsilon^2}{2(m-2)\, m\,(m+1)} |w|^{4-2m} \right) \\
&+ \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \mathcal{O}\left(r_\varepsilon^{2m-4} |w|^{4-2m} \left(\tilde{\phi}_2 + \tilde{\phi}_4 \right) + r_\varepsilon^{5-2m} |w|^{5-2m} \sum_{h=1}^{K} \tilde{\phi}_{2h+1} \right) .
\end{aligned}
$$

Expanding and reordering terms and recalling that $\hat{b}_j = \sqrt[2m]{\bar{b}_j^{2m} + \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \varepsilon^{-2m}}$ we have

$$
\begin{aligned}
\Psi^{o}_{\mathbf{b},\mathbf{hk}}\left(r_\varepsilon w\right) =& \frac{r_\varepsilon^2 |w|^2}{2} + \psi_g\left(r_\varepsilon w\right) \\
&+ c_{\Gamma_j} \hat{b}_j^{2m} \varepsilon^{2m} r_\varepsilon^{2-2m} \left(-|w|^{2-2m} + \frac{(m-1)\, s_g r_\varepsilon^2}{2\,(m-2)\, m\,(m+1)} |w|^{4-2m} \right) \\
&+ \varepsilon^2 \hat{b}_j^2 \psi_{\eta_j} \left(\frac{r_\varepsilon w}{\varepsilon \hat{b}_j} \right) \\
&+ \varepsilon^2 \left[\bar{b}_j^2 \psi_{\eta_j} \left(\frac{r_\varepsilon w}{\varepsilon \bar{b}_j} \right) - \hat{b}_j^2 \psi_{\eta_j} \left(\frac{r_\varepsilon w}{\varepsilon \hat{b}_j} \right) \right] \\
&+ \varepsilon^{2m} r_\varepsilon^{4-2m} |w|^{4-2m} \tilde{\phi}_2 + \varepsilon^{2m} r_\varepsilon^{4-2m} |w|^{4-2m} \tilde{\phi}_4 \\
&+ \varepsilon^{2m} r_\varepsilon^{5-2m} |w|^{5-2m} \sum_{h=1}^{K} \tilde{\phi}_{2h+1} + \mathcal{O}\left(\varepsilon^{2m} r_\varepsilon^{6-2m} |w|^{6-2m} \right) \\
&+ H^{o}_{h_j k_j}\left(w\right) \\
&+ \tilde{f}^{o}_{\mathbf{b},\mathbf{hk}}\left(r_\varepsilon w\right) \\
&+ \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \mathcal{O}\left(r_\varepsilon^{2m-4} |w|^{4-2m} \left(\tilde{\phi}_2 + \tilde{\phi}_4 \right) + r_\varepsilon^{5-2m} |w|^{5-2m} \sum_{h=1}^{K} \tilde{\phi}_{2h+1} \right) .
\end{aligned}
$$

We make a further refinement of the analysis of the structure of $\Psi^{o}_{\mathbf{b},\mathbf{hk}}$: we separate its radial part from its non-radial part.

- The radial part

$$
\begin{aligned}
\left(\Psi^{o}_{\mathbf{b},\mathbf{hk}}\right)^{(0)}\left(r_\varepsilon w\right) =& \frac{r_\varepsilon^2 |w|^2}{2} + \psi_g^{(0)}\left(r_\varepsilon w\right) \\
&+ c_{\Gamma_j} \hat{b}_j^{2m} \varepsilon^{2m} r_\varepsilon^{2-2m} \left(-|w|^{2-2m} + \frac{(m-1)\, s_g r_\varepsilon^2}{2\,(m-2)\, m\,(m+1)} |w|^{4-2m} \right) \\
&+ \varepsilon^2 \hat{b}_j^2 \psi^{(0)}_{\eta_j} \left(\frac{r_\varepsilon w}{\varepsilon \hat{b}_j} \right) \\
&+ \varepsilon^2 \left[\bar{b}_j^2 \psi_{\eta_j} \left(\frac{r_\varepsilon w}{\varepsilon \bar{b}_j} \right) - \hat{b}_j^2 \psi_{\eta_j} \left(\frac{r_\varepsilon w}{\varepsilon \hat{b}_j} \right) \right]^{(0)} + \mathcal{O}\left(\varepsilon^{2m} r_\varepsilon^{6-2m} |w|^{6-2m} \right) \\
&+ \left(h_j^{(0)} - \frac{k_j^{(0)}}{4\,(m-2)} \right) |w|^{2-2m} + \frac{k_j^{(0)}}{4\,(m-2)} |w|^{4-2m} \\
&+ \left(\tilde{f}^{o}_{\mathbf{b},\mathbf{hk}} \right)^{(0)} \left(r_\varepsilon w\right) + \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \mathcal{O}\left(r_\varepsilon^{6-2m} |w|^{6-2m} \right) .
\end{aligned}
$$

Again we can collect some terms defining

$$
\begin{aligned}
\left(\zeta_j^{o}\right)^{(0)}\left(w\right) :=& \varepsilon^2 \left[\bar{b}_j^2 \psi_{\eta_j} \left(\frac{r_\varepsilon w}{\varepsilon \bar{b}_j} \right) - \hat{b}_j^2 \psi_{\eta_j} \left(\frac{r_\varepsilon w}{\varepsilon \hat{b}_j} \right) \right]^{(0)} \\
&+ \mathcal{O}\left(\varepsilon^{2m} r_\varepsilon^{6-2m} |w|^{6-2m} \right) \\
&+ \left(\tilde{f}^{o}_{\mathbf{b},\mathbf{hk}} \right)^{(0)} \left(r_\varepsilon w\right) + \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}} \mathcal{O}\left(r_\varepsilon^{6-2m} |w|^{6-2m} \right)
\end{aligned}
$$

and so we can write

$$
\begin{aligned}
\left(\Psi^o_{\mathbf{b},\mathbf{hk}}\right)^{(0)}(r_\varepsilon w) = & \frac{r_\varepsilon^2|w|^2}{2} + \psi_g^{(0)}(r_\varepsilon w) \\
& + c_{\Gamma_j}\hat{b}_j^{2m}\varepsilon^{2m}r_\varepsilon^{2-2m}\left(-|w|^{2-2m} + \frac{(m-1)\,s_g r_\varepsilon^2}{2\,(m-2)\,m\,(m+1)}|w|^{4-2m}\right) \\
& + \varepsilon^2\hat{b}_j^2\psi_{\eta_j}^{(0)}\left(\frac{r_\varepsilon w}{\varepsilon\hat{b}_j}\right) \\
& + \left(h_j^{(0)} - \frac{k_j^{(0)}}{4(m-2)}\right)|w|^{2-2m} + \frac{k_j^{(0)}}{4(m-2)}|w|^{4-2m} \\
& + \left(\zeta_j^o\right)^{(0)}(w)\,.
\end{aligned}
$$

Again, the green terms are matched perfectly with their counterpart on X_j and we will have to deal only with terms written in black.

- The non-radial part

$$
\begin{aligned}
\left(\Psi^o_{\mathbf{b},\mathbf{hk}}\right)^{(\dagger)}(r_\varepsilon w) = & \psi_g^{(\dagger)}(r_\varepsilon w) \\
& + \varepsilon^2\hat{b}_j^2\psi_{\eta_j}^{(\dagger)}\left(\frac{r_\varepsilon w}{\varepsilon\bar{b}_j}\right) \\
& + \varepsilon^2\left[\bar{b}_j^2\psi_{\eta_j}\left(\frac{r_\varepsilon w}{\varepsilon\bar{b}_j}\right) - \hat{b}_j^2\psi_{\eta_j}\left(\frac{r_\varepsilon w}{\varepsilon\hat{b}_j}\right)\right]^{(\dagger)} \\
& + \varepsilon^{2m}r_\varepsilon^{4-2m}|w|^{4-2m}\tilde{\phi}_2 + \varepsilon^{2m}r_\varepsilon^{4-2m}|w|^{4-2m}\tilde{\phi}_2 \\
& + \varepsilon^{2m}r_\varepsilon^{5-2m}|w|^{5-2m}\sum_{h=1}^{K}\tilde{\phi}_{2h+1} + \mathcal{O}\left(\varepsilon^{2m}r_\varepsilon^{6-2m}|w|^{6-2m}\right) \\
& + H^o_{h_j^{(\dagger)}k_j^{(\dagger)}}(w) \\
& + \left(\tilde{f}^o_{\mathbf{b},\mathbf{hk}}\right)^{(\dagger)}(r_\varepsilon w) \\
& + \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}}\mathcal{O}\left(r_\varepsilon^{2m-4}|w|^{4-2m}\left(\tilde{\phi}_2 + \tilde{\phi}_4\right) + r_\varepsilon^{5-2m}|w|^{5-2m}\sum_{h=1}^{K}\tilde{\phi}_{2h+1}\right)\,.
\end{aligned}
$$

Defining

$$
\begin{aligned}
\left(\zeta_j^o\right)^{(\dagger)}(w) := & \varepsilon^2\left[\bar{b}_j^2\psi_{\eta_j}\left(\frac{r_\varepsilon w}{\varepsilon\bar{b}_j}\right) - \hat{b}_j^2\psi_{\eta_j}\left(\frac{r_\varepsilon w}{\varepsilon\hat{b}_j}\right)\right]^{(\dagger)} \\
& + \varepsilon^{2m}r_\varepsilon^{4-2m}|w|^{4-2m}\tilde{\phi}_2 + \varepsilon^{2m}r_\varepsilon^{4-2m}|w|^{4-2m}\tilde{\phi}_2 \\
& + \varepsilon^{2m}r_\varepsilon^{5-2m}|w|^{5-2m}\sum_{h=1}^{K}\tilde{\phi}_{2h+1} + \mathcal{O}\left(r_\varepsilon^{2+\beta}|w|^{6-2m}\right) \\
& + \left(\tilde{f}^o_{\mathbf{b},\mathbf{hk}}\right)^{(\dagger)}(r_\varepsilon w) + \hat{f}^{o,j}_{\mathbf{b},\mathbf{hk}}\mathcal{O}\left(r_\varepsilon^{2m-4}|w|^{4-2m}\left(\tilde{\phi}_2 + \tilde{\phi}_4\right)\right)
\end{aligned}
$$

we can write

$$\left(\Psi^{o}_{\mathbf{b},\mathbf{hk}}\right)^{(\dagger)}(r_{\varepsilon}w) = \psi_{g}^{(\dagger)}(r_{\varepsilon}w) + \varepsilon^{2}\hat{b}_{j}^{2}\psi_{\eta_j}^{(\dagger)}\left(\frac{r_{\varepsilon}w}{\varepsilon\hat{b}_{j}}\right) + H^{o}_{h_j^{(\dagger)}k_j^{(\dagger)}}(w) + \left(\zeta_{j}^{o}\right)^{(\dagger)}(w)\,.$$

Again, the red terms are matched perfectly with their counterpart on X_j and we will have to deal only with terms written in black.

4.1.3 Matching at the boundaries

We want that the functions $\Psi^{o}_{\mathbf{b},\mathbf{hk}}(r_{\varepsilon}\cdot) \in C^{4,\alpha}(B_2 \setminus B_1)$ and $\Psi^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}\left(\frac{R_{\varepsilon}}{\hat{b}_j}\cdot\right) \in C^{4,\alpha}\left(B_1 \setminus B_{1/2}\right)$ match at ∂B_1 to give functions $\Psi_j \in C^{4,\alpha}(B_2 \setminus B_{1/2})$

$$\Psi_j(w) := \begin{cases} \Psi^{o}_{\mathbf{b},\mathbf{hk}}(r_{\varepsilon}w) & w \in B_2 \setminus B_1 \\ \Psi^{I}_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}\left(\frac{R_{\varepsilon}}{\hat{b}_j}w\right) & w \in B_1 \setminus B_{1/2} \end{cases}$$

Remark 4.1. It is a well known fact that given two functions $f_1 \in C^{4,\alpha}\left(B_1 \setminus B_{1/2}\right)$ and $f_2 \in C^{4,\alpha}(B_2 \setminus B_1)$ they glue to a $f \in C^{4,\alpha}\left(B_2 \setminus B_{1/2}\right)$

$$f(w) := \begin{cases} f_1(w) & w \in B_1 \setminus B_{1/2} \\ f_2(w) & w \in B_2 \setminus B_1 \end{cases}$$

if and only if at ∂B_1 the following identities hold

$$\begin{cases} f_1 &= f_2 \\ \partial_{|w|}f_1 &= \partial_{|w|}f_2 \\ \Delta f_1 &= \Delta f_2 \\ \partial_{|w|}\Delta f_1 &= \partial_{|w|}\Delta f_2 \end{cases}$$

The above remark allows us to formulate the matching problem: we want to solve on ∂B_1 the following system of PDEs

$$(\Sigma): \begin{cases} H^{o}_{h_jk_j} + \left(\zeta_j^{o}\right)^{(0)} + \left(\zeta_j^{o}\right)^{(\dagger)} &= H^{I}_{\varepsilon^2\tilde{h}_j\varepsilon^2\tilde{k}_j} + \left(\zeta_j^{I}\right)^{(0)} + \left(\zeta_j^{I}\right)^{(\dagger)} \\ \partial_{|w|}\left[H^{o}_{h_jk_j} + \left(\zeta_j^{o}\right)^{(0)} + \left(\zeta_j^{o}\right)^{(\dagger)}\right] &= \partial_{|w|}\left[H^{I}_{\varepsilon^2\tilde{h}_j\varepsilon^2\tilde{k}_j} + \left(\zeta_j^{I}\right)^{(0)} + \left(\zeta_j^{I}\right)^{(\dagger)}\right] \\ \Delta\left[H^{o}_{h_jk_j} + \left(\zeta_j^{o}\right)^{(0)} + \left(\zeta_j^{o}\right)^{(\dagger)}\right] &= \Delta\left[H^{I}_{\varepsilon^2\tilde{h}_j\varepsilon^2\tilde{k}_j} + \left(\zeta_j^{I}\right)^{(0)} + \left(\zeta_j^{I}\right)^{(\dagger)}\right] \\ \partial_{|w|}\Delta\left[H^{o}_{h_jk_j} + \left(\zeta_j^{o}\right)^{(0)} + \left(\zeta_j^{o}\right)^{(\dagger)}\right] &= \partial_{|w|}\Delta\left[H^{I}_{\varepsilon^2\tilde{h}_j\varepsilon^2\tilde{k}_j} + \left(\zeta_j^{I}\right)^{(0)} + \left(\zeta_j^{I}\right)^{(\dagger)}\right] \end{cases}$$

Since (Σ) is defined on $\partial B_1(0)$ we have

$$\begin{cases} h_j + \left(\zeta_j^{o}\right)^{(0)} + \left(\zeta_j^{o}\right)^{(\dagger)} &= \varepsilon^2\tilde{h}_j + \left(\zeta_j^{I}\right)^{(0)} + \left(\zeta_j^{I}\right)^{(\dagger)} \\ \partial_{|w|}\left[H^{o}_{h_jk_j} + \left(\zeta_j^{o}\right)^{(0)} + \left(\zeta_j^{o}\right)^{(\dagger)}\right] &= \partial_{|w|}\left[H^{I}_{\varepsilon^2\tilde{h}_j\varepsilon^2\tilde{k}_j} + \left(\zeta_j^{I}\right)^{(0)} + \left(\zeta_j^{I}\right)^{(\dagger)}\right] \\ k_j + \Delta\left[\left(\zeta_j^{o}\right)^{(0)} + \left(\zeta_j^{o}\right)^{(\dagger)}\right] &= \varepsilon^2\tilde{k}_j + \Delta\left[\left(\zeta_j^{I}\right)^{(0)} + \left(\zeta_j^{I}\right)^{(\dagger)}\right] \\ \partial_{|w|}\Delta\left[H^{o}_{h_jk_j} + \left(\zeta_j^{o}\right)^{(0)} + \left(\zeta_j^{o}\right)^{(\dagger)}\right] &= \partial_{|w|}\Delta\left[H^{I}_{\varepsilon^2\tilde{h}_j\varepsilon^2\tilde{k}_j} + \left(\zeta_j^{I}\right)^{(0)} + \left(\zeta_j^{I}\right)^{(\dagger)}\right] \end{cases}$$

We find the relations

$$\varepsilon^2 \tilde{h}_j = h_j + \left(\zeta_j^o\right)^{(0)} - \left(\zeta_j^I\right)^{(0)} + \left(\zeta_j^o\right)^{(\dagger)} - \left(\zeta_j^I\right)^{(\dagger)}$$
$$\varepsilon^2 \tilde{k}_j = k_j + \Delta\left[\left(\zeta_j^o\right)^{(0)} - \left(\zeta_j^I\right)^{(0)} + \left(\zeta_j^o\right)^{(\dagger)} - \left(\zeta_j^I\right)^{(\dagger)}\right]$$

Moreover we define

$$\xi_j^{(0)} := \left(\zeta_j^o\right)^{(0)} - \left(\zeta_j^I\right)^{(0)} \tag{4.1}$$
$$\xi_j^{(\dagger)} := \left(\zeta_j^o\right)^{(\dagger)} - \left(\zeta_j^I\right)^{(\dagger)} \tag{4.2}$$

and substituting in (Σ) we obtain the equations

$$\begin{cases} \varepsilon^2 \tilde{h}_j & = & h_j + \xi_j^{(0)} + \xi_j^{(\dagger)} \\ \partial_{|w|} H^o_{h_j k_j} & = & \partial_{|w|} H^I_{h_j k_j} + \partial_{|w|} H^I_{\xi_j^{(0)}, \Delta\xi_j^{(0)}} + \partial_{|w|} H^I_{\xi_j^{(\dagger)}, \Delta\xi_j^{(\dagger)}} + \partial_{|w|}\left[\xi_j^{(0)} + \xi_j^{(\dagger)}\right] \\ \varepsilon^2 \tilde{k}_j & = & k_j + \Delta\left[\xi_j^{(0)} + \xi_j^{(\dagger)}\right] \\ \partial_{|w|} \Delta H^o_{h_j k_j} & = & \partial_{|w|} \Delta H^I_{h_j k_j} + \partial_{|w|} \Delta H^I_{\xi_j^{(0)}, \Delta\xi_j^{(0)}} + \partial_{|w|} \Delta H^I_{\xi_j^{(\dagger)}, \Delta\xi_j^{(\dagger)}} + \partial_{|w|}\Delta\left[\xi_j^{(0)} + \xi_j^{(\dagger)}\right] \end{cases}$$

Remark 4.2. In the second and fourth equation we used the fact that by their very definition $H^o_{h,k}, H^I_{h,k}$ are (continuous) linear operators

$$H^I_{\cdot,\cdot} : \mathcal{B}_\alpha \to C^{4,\alpha}\left(B_1\right)\,,$$
$$H^o_{\cdot,\cdot} : \mathcal{B}_\alpha \to C^{4,\alpha}\left(\mathbb{C}^m \setminus B_1\right)\,.$$

The second and the fourth equations above give us the relations

$$\partial_{|w|} H^o_{h_j k_j} - \partial_{|w|} H^I_{h_j k_j} = \partial_{|w|} H^I_{\xi_j^{(0)}, \Delta\xi_j^{(0)}} + \partial_{|w|} H^I_{\xi_j^{(\dagger)}, \Delta\xi_j^{(\dagger)}} + \partial_{|w|}\left[\xi_j^{(0)} + \xi_j^{(\dagger)}\right]\,,$$
$$\partial_{|w|} \Delta H^o_{h_j k_j} - \partial_{|w|} \Delta H^I_{h_j k_j} = \partial_{|w|} \Delta H^I_{\xi_j^{(0)}, \Delta\xi_j^{(0)}} + \partial_{|w|} \Delta H^I_{\xi_j^{(\dagger)}, \Delta\xi_j^{(\dagger)}} + \partial_{|w|}\Delta\left[\xi_j^{(0)} + \xi_j^{(\dagger)}\right]\,.$$

On ∂B_1 we can rewrite the system (Σ) as

$$\begin{cases} \varepsilon^2 \tilde{h}_j & = & h_j + \xi_j^{(0)} + \xi_j^{(\dagger)} \\ \partial_{|w|} H^o_{h_j k_j} - \partial_{|w|} H^I_{h_j k_j} & = & \partial_{|w|} H^I_{\xi_j^{(0)}, \Delta\xi_j^{(0)}} + \partial_{|w|} H^I_{\xi_j^{(\dagger)}, \Delta\xi_j^{(\dagger)}} + \partial_{|w|}\left[\xi_j^{(0)} + \xi_j^{(\dagger)}\right] \\ \varepsilon^2 \tilde{k}_j & = & k_j + \Delta\left[\xi_j^{(0)} + \xi_j^{(\dagger)}\right] \\ \partial_{|w|} \Delta H^o_{h_j k_j} - \partial_{|w|} \Delta H^I_{h_j k_j} & = & \partial_{|w|} \Delta H^I_{\xi_j^{(0)}, \Delta\xi_j^{(0)}} + \partial_{|w|} \Delta H^I_{\xi_j^{(\dagger)}, \Delta\xi_j^{(\dagger)}} + \partial_{|w|}\Delta\left[\xi_j^{(0)} + \xi_j^{(\dagger)}\right] \end{cases}$$

4.2 Conclusion of the proof of Theorem 1.7

We start this section with a remark.

Remark 4.3. The differential operators $\partial_{|w|}, \Delta$ commute with the $\Delta_{S^{2m-1}}$-eigenfunction decomposition of any $f_1 \in C^{4,\alpha}\left(B_1 \setminus B_{1/2}\right)$ and $f_2 \in C^{4,\alpha}\left(B_2 \setminus B_1\right)$,indeed, for $n = 1, 2$

$$f_n(w) = \sum_{k=0}^{+\infty} (f_n)^{(k)}(|w|)\, \phi_k\left(\frac{w}{|w|}\right)$$

then

$$\partial_{|w|} f_n(w) = \sum_{k=0}^{+\infty} \partial_{|w|} (f_n)^{(k)}(|w|)\, \phi_k\left(\frac{w}{|w|}\right),$$

$$\Delta f_n(w) = \sum_{k=0}^{+\infty} \left[\partial^2_{|w|} + \frac{(2m-1)}{|w|}\partial_{|w|} - \frac{k(2m-2+k)}{|w|^2}\right]\left((f_n)^{(k)}(|w|)\right)\phi_k\left(\frac{w}{|w|}\right).$$

To conclude the gluing process we will use the fact that the "Dirichlet to Neumann" map is an isomorphism. We have indeed the following result whose proof can be found in [AP06, Lemma 6.3].

Theorem 4.1. *The map*

$$\mathcal{P} : C^{4,\alpha}(\partial B_1) \times C^{2,\alpha}(\partial B_1) \to C^{3,\alpha}(\partial B_1) \times C^{1,\alpha}(\partial B_1)$$
$$\mathcal{P}(h,k) = \left(\partial_{|w|}\left(H^o_{h,k} - H^I_{h,k}\right), \partial_{|w|}\Delta\left(H^o_{h,k} - H^I_{h,k}\right)\right)$$

is an isomorphism of Banach spaces.

Proof. The inverse $\mathcal{Q}$ of $\mathcal{P}$

$$\mathcal{Q} : C^{3,\alpha}(\partial B_1) \times C^{1,\alpha}(\partial B_1) \to C^{4,\alpha}(\partial B_1) \times C^{2,\alpha}(\partial B_1)$$

is given by the following formulas

$$\begin{aligned}\mathcal{Q}(e,f) &= \left(\mathcal{Q}_1(e,f), \mathcal{Q}_2(e,f)\right)\\ &:= \left(\sum_{\gamma=0}^{+\infty} \frac{1}{2(\gamma+m-1)}\left[e^{(\gamma)} - \frac{f^{(\gamma)}}{2(\gamma+m-2)(\gamma+m)}\right]\phi_\gamma, \sum_{\gamma=0}^{+\infty} \frac{f^{(\gamma)}}{2(\gamma+m-1)}\phi_\gamma\right)\end{aligned}$$

and it is not hard to check that $\mathcal{Q}$ is continuous. □

Using Theorem 4.1 we can define a continuous nonlinear differential operator

$$\mathcal{S} : \mathfrak{B}(\kappa,\beta,\sigma)^2 \to \mathcal{B}^2_\alpha$$
$$\mathcal{S} := \left(\mathcal{S}_{1j}, \mathcal{S}_{2j}, \mathcal{S}_{3j}, \mathcal{S}_{4j}\right) \qquad 1 \le j \le N$$

that decomposes in two pieces:

$$\begin{aligned}\mathcal{S} &= \mathcal{S}^{(0)} + \mathcal{S}^{(\dagger)}\\ &= \left(\mathcal{S}^{(0)}_{1j} + \mathcal{S}^{(\dagger)}_{1j}, \mathcal{S}^{(0)}_{2j} + \mathcal{S}^{(\dagger)}_{2j}, \mathcal{S}^{(0)}_{3j} + \mathcal{S}^{(\dagger)}_{3j}, \mathcal{S}^{(0)}_{4j} + \mathcal{S}^{(\dagger)}_{4j}\right)\end{aligned}$$

that are

$$\mathcal{S}^{(0)}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right) = \begin{cases} \mathcal{S}^{(0)}_{1j}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right) = h^{(0)}_j + \xi^{(0)}_j \\ \mathcal{S}^{(0)}_{2j}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right) = k^{(0)}_j + \Delta\xi^{(0)}_j \\ \mathcal{S}^{(0)}_{3j}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right) = \mathcal{Q}_1\left[\partial_{|w|} H^I_{\xi^{(0)}_j, \Delta\xi^{(0)}_j} + \partial_{|w|}\xi^{(0)}_j\right] \\ \mathcal{S}^{(0)}_{4j}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right) = \mathcal{Q}_2\left[\partial_{|w|}\Delta H^I_{\xi^{(0)}_j, \Delta\xi^{(0)}_j} + \partial_{|w|}\Delta\xi^{(0)}_j\right] \end{cases} \tag{4.3}$$

$$\mathcal{S}^{(\dagger)}\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}},\mathbf{h},\mathbf{k}\right)=\begin{cases}\mathcal{S}^{(\dagger)}_{1j}\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}},\mathbf{h},\mathbf{k}\right)=h_j^{(\dagger)}+\xi_j^{(\dagger)}\\ \mathcal{S}^{(\dagger)}_{2j}\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}},\mathbf{h},\mathbf{k}\right)=k_j^{(\dagger)}+\Delta\xi_j^{(\dagger)}\\ \mathcal{S}^{(\dagger)}_{3j}\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}},\mathbf{h},\mathbf{k}\right)=\mathcal{Q}_1\left[\partial_{|w|}H^I_{\xi_j^{(\dagger)},\Delta\xi_j^{(\dagger)}}+\partial_{|w|}\xi_j^{(\dagger)}\right]\\ \mathcal{S}^{(\dagger)}_{4j}\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}},\mathbf{h},\mathbf{k}\right)=\mathcal{Q}_2\left[\partial_{|w|}\Delta H^I_{\xi_j^{(\dagger)},\Delta\xi_j^{(\dagger)}}+\partial_{|w|}\Delta\xi_j^{(\dagger)}\right]\end{cases}\tag{4.4}$$

We want to find a fixed point for $\mathcal{S}$: functions $\mathbf{h},\mathbf{k},\tilde{\mathbf{h}},\tilde{\mathbf{k}}$ such that

$$\mathcal{S}\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}},\mathbf{h},\mathbf{k}\right)=\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}},\mathbf{h},\mathbf{k}\right).$$

We can see that, by the very definition of $\mathcal{S}$, fixed points are the boundary conditions we need to conclude the gluing procedure. We will find a fixed point showing that a sequence of type

$$\left(\varepsilon^2\tilde{\mathbf{h}}_{n+1},\varepsilon^2\tilde{\mathbf{k}}_{n+1},\mathbf{h}_{n+1},\mathbf{k}_{n+1}\right)=\mathcal{S}\left(\varepsilon^2\tilde{\mathbf{h}}_n,\varepsilon^2\tilde{\mathbf{k}}_n,\mathbf{h}_n,\mathbf{k}_n\right)$$

with $\left(\tilde{\mathbf{h}}_0,\tilde{\mathbf{k}}_0,\mathbf{h}_0,\mathbf{k}_0\right)=(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$ is confined in a certain bounded subset of $\mathcal{B}_\alpha$. To go on with this argument we have to be careful and carry on two different analysis: one for $\mathcal{S}^{(0)}$ and one for $\mathcal{S}^{(\dagger)}$. We now want to estimate

$$\left\|\mathcal{S}\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}},\mathbf{h},\mathbf{k}\right)\right\|_{\mathcal{B}^2_\alpha}$$

In the previous arguments we picked functions

$$\left(\varepsilon^2\tilde{\mathbf{h}},\varepsilon^2\tilde{\mathbf{k}},\mathbf{h},\mathbf{k}\right)\in\mathfrak{B}\left(\kappa,\beta,\sigma\right)^2$$

and now we want to find a $\kappa>0$ depending only on g,η_j such that the operator $\mathcal{S}$ has, on $\mathfrak{B}\left(\kappa,\beta,\sigma\right)^2$, some sort of compactness.

Proposition 4.1. *There exist $\kappa>0$ depending only on g,η_j and not on ε such that*

$$\mathcal{S}:\mathfrak{B}\left(\kappa,\beta,\sigma\right)^2\to\mathfrak{B}\left(\frac{3}{2}\kappa,\beta,\sigma\right)\times\mathfrak{B}\left(\kappa,\beta,\sigma\right).$$

Proof. The thesis will follow from estimates on $\mathcal{S}$ and we recall its definition given by formulas (4.3) and (4.4). To have estimates on $\mathcal{S}$ we need only to get estimates on functions $\xi_j^{(0)},\xi_j^{(\dagger)}$ defined by formulas (4.1) and (4.2). Making use of:

- expansions (2.1),(2.2) of the blow up asymptotics of functions in the deficiency space $\mathcal{D}$,
- formulæ(3.1),(3.3) defining skeletons on the orbifold and on ALE spaces,
- formula (3.4) defining modified biharmonic extensions on the ALE spaces,
- Proposition 3.4 and Proposition 3.5 that give us estimates on $f^o_{\mathbf{b},\mathbf{hk}}$ and $f^I_{\tilde{h}_j\tilde{k}_j}$,
- the choice of parameters $\lambda,\beta,\sigma,\mu,\mu',\tau,\delta$

we have that

$$\left\|\xi_j^{(0)}\right\|_{C^{4,\alpha}(\partial B_1)} \leq C\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\beta \tag{4.5}$$

and

$$\left\|\xi_j^{(\dagger)}\right\|_{C^{4,\alpha}(\partial B_1)} \leq C\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\sigma \,. \tag{4.6}$$

From the above inequalities we can immediately obtain the following estimates on $\mathcal{S}$

$$\begin{cases} \left|\mathcal{S}_{1j}^{(0)}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right)\right| \leq \left|h_j^{(0)}\right| + C_1\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\beta \\ \left|\mathcal{S}_{2j}^{(0)}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right)\right| \leq \left|k_j^{(0)}\right| + C_2\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\beta \\ \left|\mathcal{S}_{3j}^{(0)}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right)\right| \leq C_3\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\beta \\ \left|\mathcal{S}_{4j}^{(0)}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right)\right| \leq C_4\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\beta \end{cases}$$

$$\begin{cases} \left\|\mathcal{S}_{1j}^{(\dagger)}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right)\right\|_{C^{4,\alpha}(\partial B_1)} \leq \left\|h_j^{(\dagger)}\right\|_{C^{4,\alpha}(\partial B_1)} + D_1\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\sigma \\ \left\|\mathcal{S}_{2j}^{(\dagger)}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right)\right\|_{C^{2,\alpha}(\partial B_1)} \leq \left\|k_j^{(\dagger)}\right\|_{C^{2,\alpha}(\partial B_1)} + D_2\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\sigma \\ \left\|\mathcal{S}_{3j}^{(\dagger)}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right)\right\|_{C^{4,\alpha}(\partial B_1)} \leq D_3\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\sigma \\ \left\|\mathcal{S}_{4j}^{(\dagger)}\left(\varepsilon^2\tilde{\mathbf{h}}, \varepsilon^2\tilde{\mathbf{k}}, \mathbf{h}, \mathbf{k}\right)\right\| \leq D_4\left(g, \boldsymbol{\eta}\right) r_\varepsilon^\sigma \end{cases}$$

The positive constants $C_1, \ldots, C_4, D_1, \ldots, D_4 \in \mathbb{R}^+$ depend only on metrics g and η_j and not on ε and κ. Let

$$C_5\left(g, \boldsymbol{\eta}\right) := \max\left\{C_1\left(g, \boldsymbol{\eta}\right), \ldots, C_4\left(g, \boldsymbol{\eta}\right), D_1\left(g, \boldsymbol{\eta}\right), \ldots, D_4\left(g, \boldsymbol{\eta}\right)\right\} \,,$$

then setting

$$\kappa = 2C_5\left(g, \boldsymbol{\eta}\right)$$

We have the thesis. □

Now we can finally prove our main result.

Theorem 1.7. *Let (M, g) be a compact cscK orbifold with isolated singularities and let*

$$\boldsymbol{p} := \{p \in M | p \text{ is a } SU(m) \text{ singularity admitting a Kähler crepant resolution}\}$$

and

$$\ker\left(\mathbb{L}_g\right) = \langle 1, \varphi_1, \ldots, \varphi_d\rangle \,.$$

Suppose moreover that

- $\sharp\boldsymbol{p} = N \geq d+1$;

- *the $d \times N$ matrix*
$$\Delta\Phi\left(\boldsymbol{p}\right)_{i,j} := \Delta_g \varphi_i\left(p_j\right) + s_g \varphi_i\left(p_j\right)$$
has full rank;

- *there exist $\boldsymbol{b} := (b_1, \ldots, b_N) \in \mathbb{R}_+^N$ such that*
$$\sum_{j=1}^{N} b_j\left[\Delta_g \varphi_i\left(p_j\right) + s_g \varphi_i\left(p_j\right)\right] = 0 \quad 1 \leq i \leq d\,.$$

Then there exist $\left(\tilde{M}, \tilde{g}_{\boldsymbol{b},\varepsilon}\right)$ cscK orbifold together with a holomorphic, surjective

$$\pi : \tilde{M} \to M\,.$$

The manifold $\tilde{M}$ is obtanied replacing $\boldsymbol{p}$ with ALE-Kähler spaces that are Ricci-flat.

Proof. We consider the sequence

$$\left(\varepsilon^2\tilde{\mathbf{h}}_{n+1}, \varepsilon^2\tilde{\mathbf{k}}_{n+1}, \mathbf{h}_{n+1}, \mathbf{k}_{n+1}\right) = \mathcal{S}\left(\varepsilon^2\tilde{\mathbf{h}}_{n}, \varepsilon^2\tilde{\mathbf{k}}_{n}, \mathbf{h}_{n}, \mathbf{k}_{n}\right)$$

with $\left(\tilde{\mathbf{h}}_0, \tilde{\mathbf{k}}_0, \mathbf{h}_0, \mathbf{k}_0\right) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$. By Proposition 4.1 we have that,

$$\begin{cases} \varepsilon^2\left|\left(\tilde{h}_j^{(0)}\right)_{n+1}\right| & \leq & \left|\left(h_j^{(0)}\right)_n\right| + \frac{\kappa}{2} r_\varepsilon^\beta \\ \varepsilon^2\left|\left(\tilde{k}_j^{(0)}\right)_{n+1}\right| & \leq & \left|\left(k_j^{(0)}\right)_n\right| + \frac{\kappa}{2} r_\varepsilon^\beta \\ \left|\left(h_j^{(0)}\right)_{n+1}\right| & \leq & \frac{\kappa}{2} r_\varepsilon^\beta \\ \left|\left(k_j^{(0)}\right)_{n+1}\right| & \leq & \frac{\kappa}{2} r_\varepsilon^\beta \end{cases}$$

$$\begin{cases} \varepsilon^2\left\|\left(\tilde{h}_j^{\dagger}\right)_{n+1}\right\|_{C^{4,\alpha}(\partial B_1)} & \leq & \left\|\left(h_j^{\dagger}\right)_n\right\|_{C^{4,\alpha}(\partial B_1)} + \frac{\kappa}{2} r_\varepsilon^\sigma \\ \varepsilon^2\left\|\left(\tilde{k}_j^{\dagger}\right)_{n+1}\right\|_{C^{2,\alpha}(\partial B_1)} & \leq & \left\|\left(k_j^{\dagger}\right)_n\right\|_{C^{2,\alpha}(\partial B_1)} + \frac{\kappa}{2} r_\varepsilon^\sigma \\ \left\|\left(h_j^{\dagger}\right)_{n+1}\right\|_{C^{4,\alpha}(\partial B_1)} & \leq & \frac{\kappa}{2} r_\varepsilon^\sigma \\ \left\|\left(k_j^{\dagger}\right)_{n+1}\right\|_{C^{2,\alpha}(\partial B_1)} & \leq & \frac{\kappa}{2} r_\varepsilon^\sigma \end{cases}$$

and hence
$$\left\{\left(\varepsilon^2\tilde{\mathbf{h}}_n, \varepsilon^2\tilde{\mathbf{k}}_n, \mathbf{h}_n, \mathbf{k}_n\right)\right\}_{n\in\mathbb{N}} \subset \mathfrak{B}\left(\kappa, \beta, \sigma\right)^2\,.$$

Since we have the compact embedding

$$\mathcal{B}_\alpha \hookrightarrow \mathcal{B}_\gamma \qquad 0<\gamma<\alpha\,,$$

we can extract a $\mathcal{B}^2_\gamma$-convergent subsequence (that by abuse of notation we call again)

$$\left\{\left(\varepsilon^2\tilde{\mathbf{h}}_n,\varepsilon^2\tilde{\mathbf{k}}_n,\mathbf{h}_n,\mathbf{k}_n\right)\right\}_{n\in\mathbb{N}}$$

to a limit

$$\left(\varepsilon^2\tilde{\mathbf{h}}_\infty,\varepsilon^2\tilde{\mathbf{k}}_\infty,\mathbf{h}_\infty,\mathbf{k}_\infty\right)\in\mathcal{B}^2_\gamma\,.$$

By construction we have that

$$\mathcal{S}\left(\varepsilon^2\tilde{\mathbf{h}}_\infty,\varepsilon^2\tilde{\mathbf{k}}_\infty,\mathbf{h}_\infty,\mathbf{k}_\infty\right)=\left(\varepsilon^2\tilde{\mathbf{h}}_\infty,\varepsilon^2\tilde{\mathbf{k}}_\infty,\mathbf{h}_\infty,\mathbf{k}_\infty\right)\,.$$

We have now found our "boundary conditions" and this implies that, by Proposition 3.4, we have a $C^{2,\gamma}\left(M_{r_\varepsilon}\right)$ cscK-metric $g_{\mathbf{b},\mathbf{hk}}$ on M_{r_ε} and , by Proposition 3.5 , $C^{2,\gamma}\left(X_{\frac{R_\varepsilon}{\hat{b}_j},j}\right)$-metrics $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ on $X_{\frac{R_\varepsilon}{\hat{b}_j},j}$ that by construction glue to a $C^{2,\gamma}\left(\tilde{M}\right)$ cscK Kähler metric $\tilde{g}_{\mathbf{b},\varepsilon}$, and by [Mor58] we have that $\tilde{g}_{\mathbf{b},\varepsilon}$ is real analytic. We have found our cscK metric on $\tilde{M}$ and therefore proved Theorem 1.7. □

Remark 4.4. We want to point out that in conditions (1.9) and (1.10) of Theorem 1.7 do not appear the orders of the groups of orbifold points as one would expect. We now explain why this phenomenon occurs. In our gluing procedure, when we construct the skeleton $\mathbb{J}_{\hat{b}}$ on an ALE space X, we "bring to X " the whole local potential ψ_g of g at the orbifold point $p\in M$ associated to X. But bringing naively ψ_g on X only cutting off and rescaling it would lead us "too far" from a cscK metric on the region of X we are interested in. Indeed we have to correct ψ_g as much as we can to "stay close" to a cscK metric. For our purposes, the most refined correction we can operate on ψ_g is adding a well chosen function containing the term

$$\frac{c_\Gamma\left(m-1\right)s_g\varepsilon^4\hat{b}^4}{2\left(m-2\right)m\left(m+1\right)}|x|^{4-2m}\,.$$

We construct, indeed, a metric $\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}$ that has has expansion at infinity

$$\omega_{\eta_{\hat{b}_j,\tilde{h}_j\tilde{k}_j}}\approx i\partial\overline{\partial}\left[\frac{\varepsilon^2\hat{b}^2|x|^2}{2}-c_\Gamma\varepsilon^2\hat{b}^2\left(|x|^{2-2m}-\frac{\left(m-1\right)s_g\varepsilon^2\hat{b}^2}{2\left(m-2\right)m\left(m+1\right)}|x|^{4-2m}\right)\right]\,.$$

To have a good matching, we want to construct, on the orbifold M, a function with the asymptotics above (suitably rescaled) near the point p. We thus want to construct a function that blows up approaching point p. Moreover, we want this function to be on $\ker\left(\mathbb{L}_g\right)$ as much as possible, so we are led to solve a PDE with distribution data. To identify the equation we have to solve we have to find out what kind of distribution is

$$\mathbb{L}_g\left(|z|^{2-2m}-\frac{\left(m-1\right)s_g}{2\left(m-2\right)m\left(m+1\right)}|z|^{4-2m}\right)\,.$$

With some efforts we discover that distributionally we have

$$\mathbb{L}_g\left(|z|^{2-2m}-\frac{\left(m-1\right)s_g}{2\left(m-2\right)m\left(m+1\right)}|z|^{4-2m}\right)=\frac{\left(m-1\right)\mu\left(S^{2m-1}\right)}{|\Gamma|}\left(\Delta_g\delta_p+s_g\delta_p\right)+\theta_p$$

with $\theta_p \in C^\infty(B_r(p))$. The equation we are interested in is

$$\mathbb{L}_g H - \frac{(m-1)\, s_g \mu\left(S^{2m-1}\right)}{\mathrm{Vol}_g(M)} \sum_{j=1}^{N} \frac{c_{\Gamma_j} b_j}{|\Gamma_j|} = -(m-1)\,\mu\left(S^{2m-1}\right) \sum_{j=1}^{N} \frac{c_{\Gamma_j} b_j}{|\Gamma_j|} \left(\Delta_g \delta_{p_j} + s_g \delta_{p_j}\right)$$

with $b_j \in \mathbb{R}^+$. That equation is solvable if and only if the linear equations

$$\sum_{j=1}^{N} \frac{c_{\Gamma_j} b_j}{\Gamma_j} \left[\Delta_g \varphi_i(p_j) + s_g \varphi_i(p_j)\right] = 0 \quad 1 \le i \le d$$

with $\{1, \varphi_1, \ldots, \varphi_d\}$ a L^2-orthogonal basis of $\ker(\mathbb{L}_g)$ are solvable. But it is immediate to see that the above equations in the b_j's are solvable if and only if there are $b'_j \in \mathbb{R}^+$ such that

$$\sum_{j=1}^{N} b'_j \left[\Delta_g \varphi_i(p_j) + s_g \varphi_i(p_j)\right] = 0 \quad 1 \le i \le d\,.$$

This shows why there aren't coefficients involving the orders of the various groups in the condition (1.10). To explain why there are no coefficients depending on quantities relative to the groups in condition (1.9) we have to recall that at a certain point we have to assure the solvability of equation

$$\mathbb{L}_g(H_f) + d_0 = \sum_{j=1}^{N} d_j \frac{c_{\Gamma_j} \mu\left(S^{2m-1}\right)}{|\Gamma_j|} \left[\Delta_g \delta_{p_j} + s_g \delta_{p_j}\right] + \sum_{l=1}^{d} f_l \varphi_l$$

for any $f_1, \ldots, f_d \in \mathbb{R}$ and $d_0, \ldots, d_N \in \mathbb{R}$ depending on f_l's. This happens if and only if the matrix

$$\left(\frac{c_{\Gamma_j} \mu\left(S^{2m-1}\right)}{|\Gamma_j|} \left[\Delta_g \varphi(p_j) + s_g \varphi(p_j)\right] \right)_{\substack{1 \le i \le d \\ 1 \le j \le N}}$$

has full rank that is equivalent to requiring the matrix

$$\left(\Delta_g \varphi(p_j) + s_g \varphi(p_j)\right)_{\substack{1 \le i \le d \\ 1 \le j \le N}}$$

to have full rank. This being said, we want to point out also that different choices of $\mathbf{b}$ give rise to different Kähler metrics on the desingularized manifold $\tilde{M}$. We have by construction, indeed, that

$$\left[\omega_{\tilde{g}_{\mathbf{b},\varepsilon}}\right] = \pi^*\left[\omega_g\right] + \sum_{j=1}^{N} \varepsilon^2 \hat{b}_j^2 \left[\tilde{\omega}_{\eta_j}\right]$$

with $\left[\tilde{\omega}_{\eta_j}\right] \in H^{(1,1)}\left(\tilde{M}, \mathbb{C}\right) \cap H^2\left(\tilde{M}, \mathbb{R}\right)$ cohomology classes such that

$$\mathfrak{i}_j^* \left[\tilde{\omega}_{\eta_j}\right] = \left[\omega_{\eta_j}\right]$$

with

$$\mathfrak{i}_j : X_{j, \frac{R_\varepsilon}{\hat{b}_j}} \hookrightarrow \tilde{M}$$

the natural embedding and

$$\lim_{\varepsilon \to 0} \hat{b}_j = \sqrt[2m]{b_j}\,.$$

4.3 $U(m)$ vs $SU(m)$

Theorem 1.7 leaves a natural question open: is there a condition mixing $SU(m)$ and $U(m)$ singular points that let us to find a cscK metric on the desingularization? Rollin and Singer in [RS09a] state without proof the following theorem that gives a partial answer to the above question. We give a proof here because this result and Theorem 1.7 are the first step in understanding if there is an “optimal” condition mixing different kind of orbifold singularities in order to get a cscK desingularization.

Theorem 4.2. *Let (M, g) be a compact cscK orbifold with isolated singularities, and let $\varphi_1, \ldots, \varphi_d \in C^\infty(M)$ such that*

$$\ker(\mathbb{L}_g) := \mathit{span}_{\mathbb{R}}\{1, \varphi_1, \ldots, \varphi_d\}\,.$$

- *Let*

  $$\boldsymbol{p} := \{p_1, \ldots, p_n\} \subseteq M \qquad n \geq d+1$$

  a set of points with negighborhoods biholomorphic to B_1/Γ_p with Γ_p a finite subgroup, even trivial, of $U(m)$. Moreover let $\mathbb{C}^m/\Gamma_p$ admit a scalar flat ALE resolution (X_p, η_p) (in the case Γ_p is trivial we consider the blow up) such that the metrics η_p have asymptotic expansion

  $$\omega_{\eta_p} = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + E_p|z|^{4-2m} + J_p|z|^{2-2m} + \mathcal{O}\left(|z|^{-2m}\right)\right) \qquad E_j \neq 0$$

  and for $m=2$

  $$\omega_{\eta_p} = i\partial\overline{\partial}\left(\frac{|z|^2}{2} + E_p\log(|z|) + J_p|z|^{-2} + \mathcal{O}\left(|z|^{-4}\right)\right) \qquad E_j \neq 0\,.$$

- *Let $\boldsymbol{q} \subseteq M$ the set of points with negighborhoods biholomorphic to B_1/Γ_q with Γ_q a finite subgroup of $SU(m)$ and such that $\mathbb{C}^m/\Gamma_q$ admit an ALE Kähler crepant resolution (Y_q, θ_q).*

  Let

$$\Phi = \left(\frac{E_{p_j}}{|E_{p_j}|}\varphi_j(p_j)\right)_{1\leq i\leq d, 1\leq j\leq n}\,.$$

If

$$rk(\Phi) = d$$

and there exist $\boldsymbol{a} := (a_1, \ldots, a_n) \in (\mathbb{R}^+)^n$ satisfying

$$\Phi\boldsymbol{a} = 0\,,$$

then there exist a compact cscK orbifold with isolated singularities $\left(\tilde{M}, \tilde{g}\right)$ with a holomorphic surjective map

$$\pi : \tilde{M} \to M$$

and $\tilde{M}$ is obtained replacing points of $\boldsymbol{p}$ with resolutions X_p and points $\boldsymbol{q}$ with resolutions Y_q.

The proof of this result is much easier than the proof of Theorem 1.7, indeed the presence of points $\mathbf{p}$ whose resolutions have the asymptotic $|x|^{4-2m}$ let us to avoid a too refined construction of skeletons solutions on M and X_j, Y_k. Moreover, in this case, we are allowed to choose a uniform size r_ε^σ for boundary conditions contrarily to the case of Theorem 1.7 where we had to choose a size r_ε^β for the radial component and a size r_ε^σ for the non radial components. Let now $\varepsilon > 0$ small and we set

$$\begin{aligned} r_\varepsilon &= \varepsilon^{\frac{2m-1}{2m+1}} \\ \sigma &= 4 \\ R_\varepsilon &= \frac{r_\varepsilon}{\varepsilon} \\ \tau, \delta &\in (0,1) \end{aligned}$$

The key step for proving Theorem 4.2 is the following application of Lemma 3.1.

Proposition 4.2. *Let (M, g) be a compact cscK orbifold with isolated singularities satisfying hypotheses of Theorem 4.2. Let $\mu_{\boldsymbol{pq}}$ a linear combination of Dirac delta functions at points in $\boldsymbol{p}$ and in $\boldsymbol{q}$ and their derivatives up to order 2. Then for ε sufficiently small there exist*

$$a_0, \dots a_n \in \mathbb{R}$$

such that equation

$$\mathbb{L}_g H^{\boldsymbol{a}} + a_0 = \sum_{i=1}^{n} \frac{E_j}{|E_j|} a_j \delta_{p_j} + \varepsilon \mu_{\boldsymbol{pq}}$$

is solvable with unique solution $H^{\boldsymbol{a}}$ orthogonal to $\ker(\mathbb{L}_g)$.

With the above proposition can construct the “skeleton” for our metric g' on M_{r_ε}.

Proposition 4.3. *Let (M, g) be a compact cscK orbifold satisfying hypotheses of Theorem 4.2. Then for any $b_q \in \mathbb{R}^+$ and*

$$h_p^{(0)}, k_p^{(0)}, k_p^{(1)}, h_q^{(0)}, k_q^{(0)} \in [-\kappa r_\varepsilon^4, \kappa r_\varepsilon^4]$$

there exist $H^{\boldsymbol{a}}_{\boldsymbol{hk}} \in C^{4,\alpha}_{4-2m,2-2m}(M_{\boldsymbol{pq}}) \cap C^\infty_{loc}(M_{\boldsymbol{pq}})$ such that

- *at $p \in \boldsymbol{p}$ has expansion*

$$\begin{aligned} H^{\boldsymbol{a}}_{\boldsymbol{hk}}(z) =& a_p \frac{E_p}{|E_p|} |z|^{4-2m} + \left(h_p^{(0)} + \frac{k_p^{(0)}}{4(m-2)} \right) r_\varepsilon^{2m-2} \varepsilon^{2-2m} |z|^{2-2m} \\ &- \frac{k_p^{(0)}}{4(m-2)} r_\varepsilon^{2m-4} \varepsilon^{2-2m} |z|^{4-2m} - \frac{k_p^{(1)}}{4(m-1)} r_\varepsilon^{2m-3} \varepsilon^{2-2m} |z|^{3-2m} \phi_1 \\ &+ \mathcal{O}\left(r_\varepsilon^{2m+2} \varepsilon^{2-2m} |z|^{4-2m} + |z|^{5-2m} \right) \end{aligned}$$

  and for $m = 2$

$$\begin{aligned} H^{\boldsymbol{a}}_{\boldsymbol{hk}}(z) =& a_p \frac{E_p}{|E_p|} \log(|z|) + \left(h_p^{(0)} + \frac{k_p^{(0)}}{2} \right) r_\varepsilon^2 \varepsilon^{-2} |z|^{-2} \\ &- \frac{k_p^{(0)}}{2} \varepsilon^{-2} \log(|z|) - \frac{k_p^{(1)}}{4} r_\varepsilon \varepsilon^{-2} |z| \phi_1 \\ & C_p + \mathcal{O}\left(r_\varepsilon^6 \varepsilon^{-2} \log(|z|) + |z| \right) \end{aligned}$$

- *at $q \in \boldsymbol{q}$ has expansion*

$$
\begin{aligned}
H^{\boldsymbol{a}}_{\boldsymbol{hk}}(z) = & - b_q \varepsilon^2 |z|^{2-2m} + \left(h^{(0)}_q + \frac{k^{(0)}_q}{4(m-2)} \right) r_\varepsilon^{2m-2} \varepsilon^{2-2m} |z|^{2-2m} \\
& - \frac{k^{(0)}_q}{4(m-2)} r_\varepsilon^{2m-4} \varepsilon^{2-2m} |z|^{4-2m} + \mathcal{O}\left(r_\varepsilon^{2m+2} \varepsilon^{2-2m} |z|^{4-2m} + |z|^{6-2m} \right)
\end{aligned}
$$

and for $m = 2$

$$
\begin{aligned}
H^{\boldsymbol{a}}_{\boldsymbol{hk}}(z) = & - b_q \varepsilon^2 |z|^{-2} + \left(h^{(0)}_q + \frac{k^{(0)}_q}{2} \right) r_\varepsilon^{2} \varepsilon^{-2} |z|^{-2} \\
& - \frac{k^{(0)}_q}{2} \varepsilon^{-2} \log(|z|) - \frac{k^{(1)}_q}{4} r_\varepsilon \varepsilon^{-2} |z| \phi_1 \\
& C_q + \mathcal{O}\left(r_\varepsilon^{6} \varepsilon^{-2} \log(|z|) + |z| \right)
\end{aligned}
$$

with $C_p, C_q \in \mathbb{R}$.

As in the proof of Theorem 1.7 we use modified biharmonic extensions $\tilde{H}^o_{\mathbf{hk}}$ that

- near points $p \in \mathbf{p}$ are

$$
\begin{aligned}
\tilde{H}^o_{\mathbf{hk}}(z) := & \chi_{p,r_0} \left[\sum_{\gamma=2}^{+\infty} \left(\left(h^{(\gamma)}_p + \frac{k^{(\gamma)}_p}{4(m+\gamma-2)} \right) \left| \frac{z}{r_\varepsilon} \right|^{2-2m-\gamma} - \frac{k^{(\gamma)}_p}{4(m+\gamma-2)} \left| \frac{z}{r_\varepsilon} \right|^{4-2m-\gamma} \right) \phi_\gamma \right] \\
& + \chi_{p,r_0} \left(h^{(1)} + \frac{k^{(1)}}{4(m-1)} \right) \left| \frac{z}{r_\varepsilon} \right|^{1-2m}
\end{aligned}
$$

- near points $q \in \mathbf{q}$ are

$$
\tilde{H}^o_{\mathbf{hk}}(z) := \chi_{q,r_0} \left[\sum_{\gamma=2}^{+\infty} \left(\left(h^{(\gamma)}_q + \frac{k^{(\gamma)}_q}{4(m+\gamma-2)} \right) \left| \frac{z}{r_\varepsilon} \right|^{2-2m-\gamma} - \frac{k^{(\gamma)}_q}{4(m+\gamma-2)} \left| \frac{z}{r_\varepsilon} \right|^{4-2m-\gamma} \right) \phi_\gamma \right]
$$

with $\chi_{p,r_0}, \chi_{q,r_0}$ cutoff functions that are identically 1 respectively on $B_{r_0}(p)$, $B_{r_0}(q)$ and identically 0 on $M \setminus B_{2r_0}(p)$, $M \setminus B_{2r_0}(q)$. So our g' on M will be again of the form

$$
\omega_{g'} = \omega_g + i\partial\overline{\partial}\left(H^{\mathbf{a}}_{\mathbf{hk}} + \tilde{H}^o_{\mathbf{hk}} + f^o_{\mathbf{b},\mathbf{hk}} \right)
$$

and $f^o_{\mathbf{b},\mathbf{hk}} \in C^{4,\alpha}_{4-2m+\tau}(M_{\mathbf{pq}}) \oplus \mathcal{D}$. The space $\mathcal{D}$, once fixed a right inverse for Φ, is the d-dimensional space of functions $H^{\mathbf{c}}$ unique solution of equation

$$
\mathbb{L}_g(H^{\mathbf{c}}) + d_0(\mathbf{c}) = \sum_{p\in\mathbf{p}} d_p(\mathbf{c}) \frac{E_p}{|E_p|} \delta_p + \sum_{j=1}^{d} c_j \varphi_j
$$

for any $\mathbf{c} \in \mathbb{R}^d$ and $d_0(\mathbf{c}), d_p(\mathbf{c}) \in \mathbb{R}$ uniquely determined by $\mathbf{c}$. For dimension $m = 2$ we need a little more care because

$$4 - 2m + \tau = \tau > 0$$

and so $C^{4,\alpha}_{\tau}(M_{\mathbf{pq}}) \subset C^{0,\alpha}(M)$. To have surjectivity for the operator

$$\mathbb{L}_g : C^{4,\alpha}_{\tau}(M_{\mathbf{pq}}) \to C^{0,\alpha}_{\tau-4}(M_{\mathbf{pq}})$$

we need to extend the space $\mathcal{D}$ with another finite dimensional space $\mathcal{D}'$ (its dimension is $\sharp\mathbf{p} + \sharp\mathbf{q}$) generated by smooth cutoff functions

$$\chi_p(x) := \begin{cases} 1 & x \in B_{r_0}(p) \\ 0 & x \in M \setminus B_{2r_0}(p) \end{cases} \qquad \chi_q(x) := \begin{cases} 1 & x \in B_{r_0}(q) \\ 0 & x \in M \setminus B_{2r_0}(q) \end{cases}$$

By abuse of notation we call $\mathcal{D}$ also the space $\mathcal{D} \oplus \mathcal{D}'$. As in chapter 3, in the very same way, we can construct a nonlinear operator $\mathcal{N}$ to find $f^o_{\mathbf{b},\mathbf{hk}}$

$$\mathcal{N} : C^{4,\alpha}_{4-2m+\tau}(M_{\mathbf{pq}}) \oplus \mathcal{D} \times \mathcal{B}_\alpha \to C^{4,\alpha}_{4-2m+\tau}(M_{\mathbf{pq}}) \oplus \mathcal{D}$$

and we have the following proposition.

Proposition 4.4. *Let $f, f' \in C^{4,\alpha}_{4-2m+\tau}(M_{\boldsymbol{pq}}) \oplus \mathcal{D}$ such that*

$$\|f\|_{C^{4,\alpha}_{4-2m+\tau}(M_{\boldsymbol{pq}}) \oplus \mathcal{D}}, \|f'\|_{C^{4,\alpha}_{4-2m+\tau}(M_{\boldsymbol{pq}}) \oplus \mathcal{D}} \leq C(g)\, \varepsilon^{4m-4} r_\varepsilon^{2-2m-\tau}.$$

Let $(\boldsymbol{h}, \boldsymbol{k}), (\boldsymbol{h}', \boldsymbol{k}') \in \mathcal{B}_\alpha$ such that

$$\|(\boldsymbol{h}, \boldsymbol{k})\|, \left\|(\boldsymbol{h}', \boldsymbol{k}')\right\| \leq \kappa r_\varepsilon^4,$$

then we have

- *for every $(\boldsymbol{h}, \boldsymbol{k})$ as above*

$$\|\mathcal{N}(0, \boldsymbol{h}, \boldsymbol{k})\|_{C^{4,\alpha}_{4-2m+\tau}(M_{\boldsymbol{pq}}) \oplus \mathcal{D}} \leq C(g)\, \varepsilon^{4m-4} r_\varepsilon^{2-2m-\tau};$$

- *for every $(\boldsymbol{h}, \boldsymbol{k})$ as above*

$$\|\mathcal{N}(f, \boldsymbol{h}, \boldsymbol{k}) - \mathcal{N}(f', \boldsymbol{h}, \boldsymbol{k})\|_{C^{4,\alpha}_{4-2m+\tau}(M_{\boldsymbol{pq}}) \oplus \mathcal{D}} \leq \frac{1}{2} \|f - f'\|_{C^{4,\alpha}_{4-2m+\tau}(M_{\boldsymbol{pq}}) \oplus \mathcal{D}};$$

- *for every f as above*

$$\left\|\mathcal{N}(f, \boldsymbol{h}, \boldsymbol{k}) - \mathcal{N}(f, \boldsymbol{h}', \boldsymbol{k}')\right\|_{C^{4,\alpha}_{4-2m+\tau}(M_{\boldsymbol{pq}}) \oplus \mathcal{D}} \leq \frac{1}{2} \left\|(\boldsymbol{h} - \boldsymbol{h}', \boldsymbol{k} - \boldsymbol{h}')\right\|.$$

The proof is identical and much easier than the analogue of chapter 3. The above proposition gives us our family of cscK metrics on M_{r_ε}.

Proposition 4.5. *Let $(\boldsymbol{h}, \boldsymbol{k}) \in \mathcal{B}_\alpha$ as above then there exist $f^o_{\boldsymbol{b},\boldsymbol{hk}} \in C^{4,\alpha}_{4+\tau-2m}(M_{\boldsymbol{p}}) \oplus \mathcal{D}$ such that*

$$\left\|f^o_{\boldsymbol{b},\boldsymbol{hk}}\right\|_{C^{4,\alpha}_{4+\tau-2m}(M_{\boldsymbol{p}}) \oplus \mathcal{D}} \leq \overline{C}(g)\, \varepsilon^{4m-4} r_\varepsilon^{2-2m-\tau}$$

and on M_{r_ε}

$$\omega_{g'} = \omega_g + i\partial\overline{\partial}\left(H^{\boldsymbol{a}}_{\boldsymbol{hk}} + \tilde{H}^o_{\boldsymbol{hk}} + f^o_{\boldsymbol{b},\boldsymbol{hk}}\right)$$

is a Kähler metric of constant scalar curvature with

$$|s_{g'} - s_g| \leq C(g)\, \varepsilon^{2m-2}.$$

Moreover the metric g' depends continuosly on $(\boldsymbol{h}, \boldsymbol{k})$.

On ALE spaces, the analysis and the construction of the families of metrics are the very same of [AP06] and [AP09] with exactly the same estimates and the same modifications for dimension $m = 2$. With the above results, the data matching procedure is the very same as [AP09] and so we prove Theorem 4.2. In complex dimension 2, Calderbank and Singer in [CS04] prove the existence of ALE scalar flat metrics on resolutions of $\mathbb{C}^2/\mathbb{Z}_k$ for any action of $\mathbb{Z}_k$ in $U(2)$ that is free on S^3. In light of this we can now apply Theorem 4.2 to the case of

- $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)/\mathbb{Z}_k$ with $k \in \mathbb{N}$ and $\mathbb{Z}_k$ acting
$$([x_0 : x_1], [y_0 : y_1]) \longrightarrow ([x_0 : \zeta_k x_1], [y_0 : \zeta_k^a y_1]) \qquad \zeta_k^k = 1 \text{ and } gcd(a,k) = 1\,;$$
- $\mathbb{P}^2/\mathbb{Z}_k$ with $k \in \mathbb{N}$ and $\mathbb{Z}_k$ acting
$$[x_0 : x_1 : x_2] \longrightarrow [x_0 : \zeta_k x_1 : \zeta_k^a x_2] \qquad \zeta_k^k = 1 \text{ and } gcd(a,k) = 1\,;$$
- $\left(\mathbb{P}^1 \times E_k\right)/\mathbb{Z}_k$ for $k = 2, 4, 6$ with E_k elliptic curve with automorphism group $\mathbb{Z}_k$ and $\mathbb{Z}_k$ acting
$$([x_0 : x_1], [z]) \longrightarrow ([x_0 : \zeta_k^a x_1], [\zeta_k z]) \qquad \zeta_k^k = 1 \text{ and } gcd(a,k) = 1\,.$$

Chapter 5

Look for examples

Now that we have Theorem 1.7 we would like to find examples where it is applicable. Once we have a cscK orbifold, if we want to apply our result there are two main difficulties:

- we have to identify $SU(m)$ singularities and if they have a crepant resolution
- we have to see if conditions on potentials of holomorphic vector fields are verified.

In general verifying these kind of hypotheses is very hard, unless we restrict to some particular kind of manifolds. We sought for examples among toric 3 dimensional orbifold which are Kähler-Einstein and Fano. Why such a choice? We restricted to

- dimension 3 because by [Roa89] for every finite subgroup $\Gamma \triangleleft SL(3, \mathbb{C})$ the singularity $\mathbb{C}^3/\Gamma$ admit a crepant resolution and if $\Gamma \triangleleft SU(3)$ the theorem of Joyce applies, so we need only to check if a singularity is $SU(3)$;
- Fano orbifolds because they have, in general, holomorphic vector fields;
- Kähler-Einstein manifolds because in this case our conditions involve only potentials of holomorphic vector fields (and so easier to verify), indeed we have that

  $$\mathbb{L}_g \varphi = \frac{1}{2} \Delta_g^2 \varphi + \frac{s_g}{2m} \Delta_g \varphi = \frac{\Delta_g}{2} \left(\Delta_g \varphi + \frac{s_g}{m} \varphi \right)$$

  and so if $\varphi \in \ker(\mathbb{L}_g)$ then it satisfies

  $$\Delta_g \varphi + \frac{s_g}{m} \varphi = 0$$

  we note moreover that our conditions in this setting become exactly conditions of Theorem 1.6;
- toric manifolds because all the above properties become combinatorial conditions on their Fans and moment polytopes.

Definition 5.1. A toric variety of dimension m is a normal variety X that contains a torus

$$\mathbb{T}^m = (\mathbb{C}^*)^m$$

as a dense open subset, together with an action

$$\mathbb{T}^m \times X \longrightarrow X$$

of $\mathbb{T}^m$ on X that extends the natural action of $\mathbb{T}^m$ on itself.

We refer to [CLS11] and [Ful93] for properties and results on toric varieties. To every m-dimensional toric variety we can associate a lattice N, its dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$ and a Fan Σ that is a union of rational polyhedral cones σ in $N \otimes_{\mathbb{Z}} \mathbb{R}$ with the following properties:

(i) If $\sigma \in \Sigma$, then $\sigma \cap (-\sigma) = \{0\}$;

(ii) If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$;

(ii) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma' \in \Sigma$.

We define $\Sigma_k \subset \Sigma$ the set of k-dimensional cones of Σ. By [CLS11] we have the following results.

Proposition 5.1. *Let X_Σ be a toric variety with fan Σ then X_Σ is an orbifold if and only if Σ is simplicial, i.e. every cone $\sigma \in \Sigma$ is a simplex.*

Proposition 5.2. *Let X_Σ be a d-dimensional toric variety with fan Σ. Let $\sigma \in \Sigma_1$, we define $\rho_\sigma \in N$ to be the minimal generator of the ray σ in N. Let $\tau \in \Sigma_d$, we define $m_\tau \in M$ to be the unique element of M s.t.*

$$\langle m_\tau, \rho_\sigma \rangle = 1$$

for every σ 1-dimensional cone of τ. Then X_Σ is Fano if and only if the set

$$P_{-K} := \{m \in M \otimes_{\mathbb{Z}} \mathbb{R} |\, \langle m, \rho_\sigma \rangle \leq 1 \text{ for } \sigma \in \Sigma_1\}$$

is a d-dimensional convex polytope whose vertices are $m_\tau \in M$ for $\tau \in \Sigma_d$.

If X_Σ is Fano say that P_{-K} is the moment polytope of X_Σ with respect the anti-canonical polarization. We now have combinatorial conditions for a toric variety to be a Fano orbifold. The Kähler-Einstein condition can be also expressed as a combinatorial condition by [WZ04], indeed the following result holds.

Theorem 5.1. *Let X_Σ a Fano toric variety, then X_Σ admit a Kähler-Einstein metric if and only if the barycenter of P_{-K} is the origin of $M \otimes_{\mathbb{Z}} \mathbb{R}$.*

We are looking for isolated $SU(3)$ singularities and because they have crepant resolutions we are looking for Fano, Kähler-Einstein toric orbifolds with canonical singularities. All these conditions can be implemented in a computer program, and in [BK13],[Kas10] such objects are completely classified. It turns out that there are 6 toric threefolds satisfying all these requests. With the help of computer program MAGMA ([BCP97]) we can determine whether a cone of the fan is singular or not and if singular which is the group of the quotient singularity.

- $X^{(1)}$ whose 1-dimensional fan $\Sigma_1^{(1)}$ is generated by points

$$\Sigma_1^{(1)} = \{(1,3,-1), (-1,0,-1), (-1,-3,1), (-1,0,0), (1,0,0), (0,0,1), (0,0,-1), (1,0,1)\}$$

and its 3-dimensional fan $\Sigma_3^{(1)}$ is generated by 12 cones

$$\begin{aligned}
C_1 &:= \langle(-1,0,-1),(-1,-3,1),(-1,0,0)\rangle\\
C_2 &:= \langle(1,3,-1),(-1,0,-1),(-1,0,0)\rangle\\
C_3 &:= \langle(-1,-3,1),(-1,0,0),(0,0,1)\rangle\\
C_4 &:= \langle(1,3,-1),(-1,0,0),(0,0,1)\rangle\\
C_5 &:= \langle(1,3,-1),(-1,0,-1),(0,0,-1)\rangle\\
C_6 &:= \langle(-1,0,-1),(-1,-3,1),(0,0,-1)\rangle\\
C_7 &:= \langle(-1,-3,1),(1,0,0),(0,0,-1)\rangle\\
C_8 &:= \langle(1,3,-1),(1,0,0),(0,0,-1)\rangle\\
C_9 &:= \langle(1,3,-1),(0,0,1),(1,0,1)\rangle\\
C_{10} &:= \langle(-1,-3,1),(1,0,0),(1,0,1)\rangle\\
C_{11} &:= \langle(1,3,-1),(1,0,0),(1,0,1)\rangle\\
C_{12} &:= \langle(-1,-3,1),(0,0,1),(1,0,1)\rangle
\end{aligned}$$

All these cones are singular and $C_1, C_4, C_5, C_7, C_{11}, C_{12}$ are cones relative to affine open subsets of $X^{(1)}$ containing a $SU(3)$ singularity, while the others are cones relative to affine open subsets of $X^{(1)}$ containing a $U(3)$ singularity.

The 3-anticanonical polytope $P_{-3K_{X^{(1)}}}$ is the convex hull of vertices

$$\begin{aligned}
P_{-3K_{X^{(1)}}} := \langle&(0,-2,-3),(-3,0,0),(-3,1,3),(0,0,3),(3,-2,0),\\
&(0,2,3),(0,0,-3),(-3,2,0),(-3,3,3),(3,0,0),(3,-1,-3),(3,-3,-3)\rangle
\end{aligned}$$

With 2-faces

$$\begin{aligned}
F_1 &:= \langle(0,-2,-3),(3,-3,-3),(-3,0,0),(-3,1,3),(0,0,3),(3,-2,0)\rangle\\
F_2 &:= \langle(-3,1,3),(0,0,3),(0,2,3),(-3,3,3)\rangle\\
F_3 &:= \langle(0,0,3),(3,-2,0),(0,2,3),(3,0,0)\rangle\\
F_4 &:= \langle(0,-2,-3),(-3,0,0),(0,0,-3),(-3,2,0)\rangle\\
F_5 &:= \langle(3,-1,-3),(0,2,3),(0,0,-3),(-3,2,0),(-3,3,3),(3,0,0)\rangle\\
F_6 &:= \langle(-3,0,0),(-3,1,3),(-3,2,0),(-3,3,3)\rangle\\
F_7 &:= \langle(3,-1,-3),(0,-2,-3),(3,-3,-3),(0,0,-3)\rangle\\
F_8 &:= \langle(3,-1,-3),(3,-3,-3),(3,-2,0),(3,0,0)\rangle
\end{aligned}$$

We have the following correspondences between cones containing a $SU(3)$-singularity and vertices of $P_{-3K_{X^{(1)}}}$

$$\begin{aligned}
C_1 &\longleftrightarrow F_3 \cap F_5 \cap F_8 = \{(3,0,0)\}\\
C_4 &\longleftrightarrow F_1 \cap F_7 \cap F_8 = \{(3,-3,-3)\}\\
C_5 &\longleftrightarrow F_1 \cap F_2 \cap F_3 = \{(0,0,3)\}\\
C_7 &\longleftrightarrow F_2 \cap F_5 \cap F_7 = \{(-3,3,3)\}\\
C_{11} &\longleftrightarrow F_1 \cap F_4 \cap F_6 = \{(-3,0,0)\}\\
C_{12} &\longleftrightarrow F_4 \cap F_5 \cap F_7 = \{(0,0,-3)\}
\end{aligned}$$

- $X^{(2)}$ whose 1-dimensional fan $\Sigma_1^{(2)}$ is generated by points

  $$\Sigma_1^{(2)} = \{(-1,2,-3),(-1,0,0),(1,0,0),(0,1,0),(1,-2,3),(-1,1,-2),(1,-1,2),(0,-1,0)\}$$

  and its 3-dimensional fan $\Sigma_3^{(2)}$ is generated by 12 cones

$$\begin{aligned}
C_1 &:= \langle(-1,2,-3),(1,0,0),(0,1,0)\rangle\\
C_2 &:= \langle(-1,2,-3),(-1,0,0),(0,1,0)\rangle\\
C_3 &:= \langle(-1,0,0),(0,1,0),(1,-2,3)\rangle\\
C_4 &:= \langle(-1,2,-3),(-1,0,0),(-1,1,-2)\rangle\\
C_5 &:= \langle(1,0,0),(1,-2,3),(1,-1,2)\rangle\\
C_6 &:= \langle(1,0,0),(0,1,0),(1,-1,2)\rangle\\
C_7 &:= \langle(0,1,0),(1,-2,3),(1,-1,2)\rangle\\
C_8 &:= \langle(-1,2,-3),(1,0,0),(0,-1,0)\rangle\\
C_9 &:= \langle(-1,0,0),(-1,1,-2),(0,-1,0)\rangle\\
C_{10} &:= \langle(-1,0,0),(1,-2,3),(0,-1,0)\rangle\\
C_{11} &:= \langle(1,0,0),(1,-2,3),(0,-1,0)\rangle\\
C_{12} &:= \langle(-1,2,-3),(-1,1,-2),(0,-1,0)\rangle
\end{aligned}$$

The cones C_1, C_{10} are relative to affine open subsets of $X^{(2)}$ containing a $SU(3)$ singularity, the cones $C_2, C_3, C_6, C_8, C_9, C_{11}$ are relative to affine open subsets of $X^{(2)}$ containing a $U(3)$ singularity and the cones C_4, C_5, C_7, C_{12} are relative to smooth affine open subsets of $X^{(2)}$. The 6-anticanonical polytope $P_{-6K_{X^{(2)}}}$ is the convex hull of vertices

$$\begin{aligned}
P_{-6K_{X^{(2)}}} := \langle&(-6,-6,0),(6,0,0),(6,-6,-4),(6,6,0),(6,-6,-8),(0,-6,-6),\\
&(-6,0,0),(-6,-6,-3),(6,6,3),(0,6,6),(-6,6,4),(-6,6,8)\rangle
\end{aligned}$$

With 2-faces

$$\begin{aligned}
F_1 &:= \langle(6,0,0),(6,-6,-4),(6,6,0),(6,-6,-8),(6,6,3)\rangle\\
F_2 &:= \langle(-6,-6,0),(-6,6,8),(6,0,0),(6,-6,-4),(0,6,6)\rangle\\
F_3 &:= \langle(-6,-6,0),(6,-6,-4),(6,-6,-8),(0,-6,-6),(-6,-6,-3)\rangle\\
F_4 &:= \langle(-6,6,4),(6,6,0),(6,-6,-8),(0,-6,-6),(-6,0,0)\rangle\\
F_5 &:= \langle(-6,6,4),(-6,-6,0),(-6,6,8),(-6,0,0),(-6,-6,-3)\rangle\\
F_6 &:= \langle(0,-6,-6),(-6,0,0),(-6,-6,-3)\rangle\\
F_7 &:= \langle(6,0,0),(6,6,3),(0,6,6)\rangle\\
F_8 &:= \langle(-6,6,4),(-6,6,8),(6,6,0),(6,6,3),(0,6,6)\rangle
\end{aligned}$$

We have the following correspondences between cones containing a $SU(3)$-singularity and vertices of $P_{-6K_{X^{(2)}}}$

$$\begin{aligned}
C_1 &\longleftrightarrow F_2 \cap F_5 \cap F_8 = \{(-6,6,,8)\}\\
C_{10} &\longleftrightarrow F_4 \cap F_5 \cap F_7 = \{(6,-6,-8)\}
\end{aligned}$$

- $X^{(3)}$ whose 1-dimensional fan $\Sigma_1^{(3)}$ is generated by points

$$\Sigma_1^{(3)} = \{(-1,0,0),(0,-1,0),(0,1,0),(1,0,0),(-1,-1,-3),(1,1,3)\}$$

and its 3-dimensional fan $\Sigma_3^{(3)}$ is generated by 8 cones

$$\begin{aligned}
C_1 &:= \langle(-1,0,0),(0,1,0),(-1,-1,-3)\rangle\\
C_2 &:= \langle(-1,0,0),(0,-1,0),(-1,-1,-3)\rangle\\
C_3 &:= \langle(0,1,0),(1,0,0),(-1,-1,-3)\rangle\\
C_4 &:= \langle(0,-1,0),(1,0,0),(-1,-1,-3)\rangle\\
C_5 &:= \langle(-1,0,0),(0,-1,0),(1,1,3)\rangle\\
C_6 &:= \langle(0,-1,0),(1,0,0),(1,1,3)\rangle\\
C_7 &:= \langle(-1,0,0),(0,1,0),(1,1,3)\rangle\\
C_8 &:= \langle(0,1,0),(1,0,0),(1,1,3)\rangle
\end{aligned}$$

The cones C_3, C_5 are relative to affine open subsets of $X^{(3)}$ containing a $SU(3)$ singularity and the other cones are relative to affine open subsets of $X^{(3)}$ containing a $U(3)$ singularity. The 3-anticanonical polytope $P_{-3K_{X^{(3)}}}$ is the convex hull of vertices

$$\begin{aligned}
P_{-3K_{X^{(3)}}} := \langle&(3,-3,1),(3,3,-1),(3,3,-3),(-3,3,1),\\
&(-3,-3,3),(-3,-3,1),(-3,3,-1),(3,-3,-1)\rangle
\end{aligned}$$

With 2-faces

$$\begin{aligned}
F_1 &:= \langle(3,3,-1(3,3,-3),(-3,3,1),(-3,3,-1)\rangle\\
F_2 &:= \langle(3,-3,1),(3,3,-1),(-3,3,1),(-3,-3,3)\rangle\\
F_3 &:= \langle(3,-3,1),(3,3,-1),(3,3,-3),(3,-3,-1)\rangle\\
F_4 &:= \langle(-3,3,1),(-3,-3,3),(-3,-3,1),(-3,3,-1)\rangle\\
F_5 &:= \langle(3,-3,1),(-3,-3,3),(-3,-3,1),(3,-3,-1)\rangle\\
F_6 &:= \langle(3,3,-3),(-3,-3,1),(-3,3,-1),(3,-3,-1)\rangle
\end{aligned}$$

We have the following correspondences between cones containing a $SU(3)$-singularity and vertices of $P_{-3K_{X^{(3)}}}$

$$\begin{aligned}
C_3 &\longleftrightarrow F_2 \cap F_4 \cap F_5 = \{(-3,-3,3)\}\\
C_5 &\longleftrightarrow F_1 \cap F_3 \cap F_6 = \{(3,3,-3)\}
\end{aligned}$$

- $X^{(4)}$ whose 1-dimensional fan $\Sigma_1^{(3)}$ is generated by points

$$\Sigma_1^{(4)} = \{(0,3,1),(1,1,2),(1,0,0),(-1,0,0),(-2,-1,-2),(1,-3,-1)\}$$

and its 3-dimensional fan $\Sigma_3^{(4)}$ is generated by 8 cones

$$
\begin{aligned}
C_1 &:= \langle (0,3,1),(1,1,2),(-1,0,0) \rangle \\
C_2 &:= \langle (0,3,1),(1,1,2),(1,0,0) \rangle \\
C_3 &:= \langle (0,3,1),(-1,0,0),(-2,-1,-2) \rangle \\
C_4 &:= \langle (0,3,1),(1,0,0),(-2,-1,-2) \rangle \\
C_5 &:= \langle (1,0,0),(-2,-1,-2),(1,-3,-1) \rangle \\
C_6 &:= \langle (1,1,2),(-1,0,0),(1,-3,-1) \rangle \\
C_7 &:= \langle (-1,0,0),(-2,-1,-2),(1,-3,-1) \rangle \\
C_8 &:= \langle (1,1,2),(1,0,0),(1,-3,-1) \rangle
\end{aligned}
$$

The cones C_1, C_4, C_7, C_8 are relative to affine open subsets of $X^{(4)}$ containing a $SU(3)$ singularity and the other cones are relative to affine open subsets of $X^{(4)}$ containing a $U(3)$ singularity.

The 5-anticanonical polytope $P_{-5K_{X^{(4)}}}$ is the convex hull of vertices

$$
\begin{aligned}
P_{-5K_{X^{(4)}}} := \langle &(5,-1,-2),(5,0,-5),(-5,-2,1),(-5,0,0), \\
&(5,5,-5),(-5,-5,10),(-5,-3,9),(5,6,-8) \rangle
\end{aligned}
$$

With 2-faces

$$
\begin{aligned}
F_1 &:= \langle (5,0,-5),(-5,-2,1),(-5,0,0),(5,6,-8) \rangle \\
F_2 &:= \langle (5,-1,-2),(5,0,-5),(-5,-2,1),(-5,-5,10) \rangle \\
F_3 &:= \langle (5,-1,-2),(5,0,-5),(5,5,-5),(5,6,-8) \rangle \\
F_4 &:= \langle (5,-1,-2),(5,5,-5),(-5,-5,10),(-5,-3,9) \rangle \\
F_5 &:= \langle (-5,-2,1),(-5,0,0),(-5,-5,10),(-5,-3,9) \rangle \\
F_6 &:= \langle (-5,0,0),(5,5,-5),(-5,-3,9),(5,6,-8) \rangle
\end{aligned}
$$

We have the following correspondences between cones containing a $SU(3)$-singularity and vertices of $P_{-5K_{X^{(4)}}}$

$$
\begin{aligned}
C_1 &\longleftrightarrow F_1 \cap F_2 \cap F_5 = \{(-5,-2,1)\} \\
C_4 &\longleftrightarrow F_2 \cap F_3 \cap F_4 = \{(5,-1,-2)\} \\
C_7 &\longleftrightarrow F_4 \cap F_5 \cap F_6 = \{(-5,-3,9)\} \\
C_8 &\longleftrightarrow F_1 \cap F_3 \cap F_6 = \{(5,6,-8)\}
\end{aligned}
$$

- $X^{(5)}$ whose 1-dimensional fan $\Sigma_1^{(5)}$ is generated by points

$$
\Sigma_1^{(5)} = \{(-1,0,0),(2,-2,-5),(1,0,0),(0,1,0),(-2,1,5)\}
$$

and its 3-dimensional fan $\Sigma_3^{(5)}$ is generated by 6 cones

$$
\begin{aligned}
C_1 &:= \langle (2,-2,-5),(1,0,0),(0,1,0)\rangle\\
C_2 &:= \langle (-1,0,0),(2,-2,-5),(0,1,0)\rangle\\
C_3 &:= \langle (2,-2,-5),(1,0,0),(-2,1,5)\rangle\\
C_4 &:= \langle (-1,0,0),(2,-2,-5),(-2,1,5)\rangle\\
C_5 &:= \langle (-1,0,0),(0,1,0),(-2,1,5)\rangle\\
C_6 &:= \langle (1,0,0),(0,1,0),(-2,1,5)\rangle
\end{aligned}
$$

The cones C_2, C_3 are relative to affine open subsets of $X^{(5)}$ containing a $SU(3)$ singularity and the other cones are relative to affine open subsets of $X^{(5)}$ containing a $U(3)$ singularity. The 5-anticanonical polytope $P_{-5K_{X^{(5)}}}$ is the convex hull of vertices

$$
P_{-5K_{X^{(5)}}} := \langle (5,-5,2),(5,10,-1),(5,-5,5),(-5,10,-5),(-5,-5,1),(-5,-5,-2)\rangle
$$

With 2-faces

$$
\begin{aligned}
F_1 &:= \langle (5,10,-1),(5,-5,5),(-5,10,-5),(-5,-5,1)\rangle\\
F_2 &:= \langle (5,-5,2),(5,10,-1),(-5,10,-5),(-5,-5,-2)\rangle\\
F_3 &:= \langle (5,-5,2),(5,10,-1),(5,-5,5)\rangle\\
F_4 &:= \langle (5,-5,2),(5,-5,5),(-5,-5,1),(-5,-5,-2)\rangle\\
F_5 &:= \langle (-5,10,-5),(-5,-5,1),(-5,-5,-2)\rangle
\end{aligned}
$$

We have the following correspondences between cones containing a $SU(3)$-singularity and vertices of $P_{-5K_{X^{(5)}}}$

$$
\begin{aligned}
C_2 &\longleftrightarrow F_1\cap F_3\cap F_4 = \{(5,-5,5)\}\\
C_3 &\longleftrightarrow F_1\cap F_2\cap F_5 = \{(-5,10,-5)\}
\end{aligned}
$$

- $X^{(6)}$ whose 1-dimensional fan $\Sigma_1^{(6)}$ is generated by points

$$
\Sigma_1^{(6)} = \{(2,-1,0),(1,3,1),(0,0,1),(-3,-2,-2)\}
$$

and its 3-dimensional fan $\Sigma_3^{(6)}$ is generated by 6 cones

$$
\begin{aligned}
C_1 &:= \langle (1,3,1),(0,0,1),(-3,-2,-2),\rangle\\
C_2 &:= \langle (2,-1,0),(0,0,1),(-3,-2,-2)\rangle\\
C_3 &:= \langle (2,-1,0),(1,3,1),(-3,-2,-2)\rangle\\
C_4 &:= \langle (2,-1,0),(1,3,1),(0,0,1)\rangle
\end{aligned}
$$

The cone C_1 is relative to affine open subsets of $X^{(6)}$ containing a $SU(3)$ singularity and the other cones are relative to affine open subsets of $X^{(6)}$ containing a $U(3)$ singularity. The 7-anticanonical polytope $P_{-7K_{X^{(6)}}}$ is the convex hull of vertices

$$
P_{-7K_{X^{(6)}}} := \langle (1,9,-7),(-3,1,-7),(9,-3,-7),(-7,-7,21)\rangle
$$

With 2-faces

$$\begin{aligned} F_1 &:= \langle(-3,1,-7),(9,-3,-7),(-7,-7,21)\rangle \\ F_2 &:= \langle(1,9,-7),(9,-3,-7),(-7,-7,21)\rangle \\ F_3 &:= \langle(1,9,-7),(-3,1,-7),(-7,-7,21)\rangle \\ F_4 &:= \langle(1,9,-7),(-3,1,-7),(9,-3,-7)\rangle \end{aligned}$$

We have the following correspondences between cones containing a $SU(3)$-singularity and vertices of $P_{-7K_{X^{(6)}}}$

$$C_1 \longleftrightarrow F_1 \cap F_2 \cap F_4 = \{(9,-3,-7)\} \ .$$

It is immediate to see that variety $X^{(6)}$ doesn't verify assumptions of Theorem 1.7 since it has only one $SU(3)$-singular point, but at the moment, we can't say if the toric orbifolds $X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}, X^{(5)}$ satisfy assumptions of Theorem 1.7 because we have no knowledge of their algebras of holomorphic vector fields vanishing somewhere.

Chapter 6

Dimension 2 and future work

6.1 Gluing in dimension 2

We proved Theorem 1.7 for orbifolds of dimension at least 3. In dimension 2, if we want to perform our gluing construction, there are many technical issues and we have to pay much more attention but, anyway, we can adapt the present construction and Theorem 1.7 extends with the same assumptions to dimension 2. In a forthcoming work we will deal with this case with all the details, but now we summarize difficulties and how key results of this thesis translate in dimension 2. The first difficulty comes from linear analysis, from the operator $\mathbb{L}_g$. Its degree is 4, the same as the real dimension of the space so its Green function has the main asymptotic that it is not a polynomial blow up, indeed it grows as $\log(z)$. The second difficulty comes from the fact that for $m = 2$ the set of indicial roots of the laplacian is the whole $\mathbb{Z}$, and so for $\delta \in (0, 1)$ we have that

$$4 - 2m + \delta = 4 - 2 \cdot 2 + \delta = \delta$$

and hence $C^{0,\alpha}_{\delta}(M_{\mathbf{p}}) \subset C^{0,\alpha}(M)$. The space $C^{0,\alpha}(M)$ is strictly greater than $C^{0,\alpha}_{\delta}(M_{\mathbf{p}})$ because it contains constant functions, so to guarantee the existence of an inverse of $\mathbb{L}_g$

$$S_\delta : C^{0,\alpha}_{\delta-4}(M_{\mathbf{p}}) \to C^{4,\alpha}_{\delta}(M_{\mathbf{p}})$$

we need to add a bigger deficiency space than in dimension 3 or higher. The inversion result for $\mathbb{L}_g$ is the following proposition.

Proposition 6.1. *Let points* $\boldsymbol{p}$ *satisfy* ***BAL(p)****, let* $\delta \in (0, 1)$ *and* $f \in C^{0,\alpha}_{\delta-4}(M_{\boldsymbol{p}})$*, let* χ_j *be smooth cutoff functions identically* 1 *on* $B_{r_0}(p_j)$ *and* 0 *on* $M \setminus B_{r_0}(p_j)$*. Then there exist* $u^{\perp} \in C^{4,\alpha}_{\delta}(M_{\boldsymbol{p}})$ *and* $H_f \in C^{\infty}_{-2}(M_{\boldsymbol{p}}) \cap C^{\infty}_{loc}(M_{\boldsymbol{p}})$ *such that, on* $M_{\boldsymbol{p}}$*,*

$$\mathbb{L}_g\left(u^{\perp} + H_f\right) + \frac{1}{Vol_g(M)} \int_M f \, d\mu_g = f$$

and

$$\left\| u^{\perp} - \sum_j u^{\perp}(p_j)\, \chi_j \right\|_{C^{4,\alpha}_{\delta}(M_p)} + \sum_j \left| u^{\perp}(p_j) \right| \leq C(g, \delta) \, \|f\|_{C^{0,\alpha}_{\delta-4}(M_p)}$$

$$\|H_f\|_{C^{4,\alpha}_{-2}(M_p)} \leq C(g, \delta) \, \|f\|_{C^{0,\alpha}_{\delta-4}(M_p)}$$

The analysis on ALE spaces in this case it's almost the same, the only thing that changes is that because of this additional deficiency space (cutoff functions) for $\mathbb{L}_g$ we need to be more careful when we construct perturbations of η_i's. Since we modify metrics η_i with the $i\partial\overline{\partial}$ of functions $\mathbb{F}_i^I$ we have to find suitable constants C_i and perturb η_i with $\mathbb{F}_i^I + C_i$. The resulting metrics are the same but their potentials are different and choosing C_i smartly we can match perfectly the coefficients of χ_i (coming from deficiency components) on M. This let us to have a better matching between the two families of metrics constructed as in chapter 3, and because of this we can perform the Cauchy data matching as in chapter 4. In complex dimension 2 we can find examples to which this gluing theory applies. The first of our examples is discussed in [RS09a] and the cscK metric on the resolution is constructed using the notion of parabolic polistability. Here we can give a direct proof of the existence of such Kähler metric.

Example 6.1. Consider $\left(\mathbb{P}^1 \times \mathbb{P}^1, \pi_1^*\omega_{FS} + \pi_2^*\omega_{FS}\right)$ and let $\mathbb{Z}_2$ act in the following way

$$([x_0 : x_1], [y_0 : y_1]) \longrightarrow ([x_0 : -x_1], [y_0 : -y_1])$$

It's immediate to check that this action is in $SU(2)$ with four fixed points

$$\begin{aligned} p_1 &= ([1:0],[1:0]) \\ p_2 &= ([1:0],[0:1]) \\ p_3 &= ([0:1],[1:0]) \\ p_4 &= ([0:1],[0:1]) \end{aligned}$$

The quotient space $X_2 := \mathbb{P}^1 \times \mathbb{P}^1/\mathbb{Z}_2$ is a Kähler-Einstein, Fano orbifold. Since it is Kähler-Einstein, conditions for applying our construction become exactly the conditions of Theorem 1.6, so we have to verify that the matrix

$$\Phi = (\varphi_i(p_j)) \qquad 1 \leq i \leq d, 1 \leq j \leq 4$$

has full rank and there exist a positive element in $\ker(\Phi)$. It is immediate to see that we have

$$H^0\left(X_2, T^{(1,0)}X_2\right) = H^0\left(\mathbb{P}^1/\mathbb{Z}_2, T^{(1,0)}\left(\mathbb{P}^1/\mathbb{Z}_2\right)\right) \oplus H^0\left(\mathbb{P}^1/\mathbb{Z}_2, T^{(1,0)}\left(\mathbb{P}^1/\mathbb{Z}_2\right)\right).$$

Moreover

$$H^0\left(\mathbb{P}^1/\mathbb{Z}_2, T^{(1,0)}\left(\mathbb{P}^1/\mathbb{Z}_2\right)\right)$$

is generated by holomorphic vector fields on $\mathbb{P}^1$ that vanish on points $[0:1], [1:0]$ so

$$\dim_{\mathbb{C}} H^0\left(\mathbb{P}^1/\mathbb{Z}_2, T^{(1,0)}\left(\mathbb{P}^1/\mathbb{Z}_2\right)\right) = 1$$

and an explicit generator is the vector field

$$V = z^1\partial_1 .$$

We can compute explicitly its potential φ_V with respect to ω_{FS} that is

$$\varphi_V([z_0 : z_1]) = \frac{1}{2}\left[\frac{|z_0 z_1|}{|z_0|^2 + |z_1|^2} + \arctan\left(\frac{|z_1|}{|z_0|}\right)\right] - \frac{3\pi}{16}$$

and it is easy to see that it is a well defined function and

$$\int_{\mathbb{P}^1} \varphi_V \omega_{FS} = 0 .$$

Summing up everything, we have that the matrix Φ for X_2 is a 2×4 matrix and can be written explicitly

$$\Phi = \frac{\pi}{16} \begin{pmatrix} -3 & -3 & 1 & 1 \\ -3 & 1 & -3 & 1 \end{pmatrix}$$

that has rank 2 and every vector of type $(a, b, b, 3a + 2b)$ for $a, b > 0$ lies in $\ker \Phi$.

Example 6.2. Consider $\left(\mathbb{P}^2, \omega_{FS}\right)$ and let $\mathbb{Z}_3$ act in the following way

$$[z_0 : z_1 : z_2] \longrightarrow [x_0 : \zeta_3 x_1 : \zeta_3^2 x_2] \qquad \zeta_3 \neq 1, \zeta_3^3 = 1$$

It's immediate to check that this action is in $SU(2)$ with three fixed points

$$p_1 = [1 : 0 : 0]$$
$$p_2 = [0 : 1 : 0]$$
$$p_3 = [0 : 0 : 1]$$

The quotient space $X_3 := \mathbb{P}^2/\mathbb{Z}_3$ is a Kähler-Einstein, Fano orbifold. Again, conditions for applying our construction become exactly the conditions of Theorem 1.6, so we have to verify that the matrix

$$\Phi = \left(\varphi_i\left(p_j\right)\right) \qquad 1 \leq i \leq d, 1 \leq j \leq 4$$

has full rank and there exist a positive element in $\ker(\Phi)$. It is immediate to see that we have

$$\dim_{\mathbb{C}} H^0\left(X_3, T^{(1,0)} X_3\right) = 2$$

because $H^0\left(X_3, T^{(1,0)} X_3\right)$ it is generated by holomorphic vector fields on $\mathbb{P}^2$ vanishing at points p_1, p_2, p_3. Explicit generators are the vector fields

$$V_1 = z^1 \partial_1 + z^2 \partial_2$$
$$V_2 = z^0 \partial_0 + z^1 \partial_1$$

We can compute explicitly their potentials $\varphi_{V_1}, \varphi_{V_2}$ with respect to ω_{FS} that are

$$\varphi_{V_1}\left([z_0 : z_1 : z_2]\right) = \frac{1}{2}\left[\frac{|z_0|\sqrt{|z_1|^2 + |z_2|^2}}{|z|^2} + \arctan\left(\frac{\sqrt{|z_1|^2 + |z_2|^2}}{|z_0|}\right)\right] - \frac{7\pi}{32}$$

$$\varphi_{V_2}\left([z_0 : z_1 : z_2]\right) = \frac{1}{2}\left[\frac{|z_2|\sqrt{|z_0|^2 + |z_1|^2}}{|z|^2} + \arctan\left(\frac{\sqrt{|z_0|^2 + |z_1|^2}}{|z_2|}\right)\right] - \frac{7\pi}{32}$$

and it is easy to see that are well defined functions and

$$\int_{\mathbb{P}^2} \varphi_{V_1} \frac{\omega_{FS}^2}{2} = \int_{\mathbb{P}^2} \varphi_{V_1} \frac{\omega_{FS}^2}{2} = 0$$

Summing up everything, we have that the matrix Φ for X_3 is a 2×3 matrix and can be written explicitly

$$\Phi = \frac{\pi}{32} \begin{pmatrix} -7 & 1 & 1 \\ 1 & 1 & -7 \end{pmatrix}$$

that has rank 2 and every vector of type $(a, 6a, a)$ for $a > 0$ lies in $\ker \Phi$.

We actually don't know what kind of rational surfaces X_2 and X_3 are.

6.2 Conjectures and future work

As we saw in chapter 5 it's difficult to find examples of cscK orbifold satisfying hypotheses of Theorem 1.7 in dimension greater or equal than 3. The main problems are the following.

- To check hypotheses of Theorem 1.7 we have to know almost explicitly the functions in $\ker \mathbb{L}_g$
- In dimension $m \geq 4$ to check which quotients $\mathbb{C}^m/\Gamma$ with $\Gamma \triangleleft SU(m)$ admit a Kähler crepant resolution.

In the future we want understand if there are more computable (cohomological or algebro-geometric) conditions on a cscK orbifold that are equivalent to hypotheses of Theorem 1.7. We conjecture that, with minor modifications, Theorem 1.7 is extendable to the case of extremal Kähler manifolds. We believe, indeed, that combining the analysis we've done in our thesis and techniques similar to [APS11] will lead us to the desired result. We also would like to study further resolutions of quotients $\mathbb{C}^m/\Gamma$ with $\Gamma \triangleleft U(m)$, indeed very little is known and the only explicit examples in dimension greater than 2 of ALE Kähler spaces that are scalar flat are $\mathcal{O}_{\mathbb{P}^m}(-k)$. We want to point out that our gluing construction relies only on the fact that the first non euclidean asymptotic of the potential of an ALE Kähler Ricci flat metric of a crepant resolution is $|z|^{2-2m}$, but with some minor modifications it would work for scalar flat ALE metrics on resolutions of $\mathbb{C}^m/\Gamma$ with $\Gamma \triangleleft U(m)$ with potentials whose first non euclidean asymptotic is $|z|^{2-2m}$. Indeed, the only difference with the Ricci-flat case is that

$$\mathbb{L}_\eta = \frac{1}{2}\Delta_\eta^2 + 2\left\langle \mathrm{Ric}_\eta, i\partial\overline{\partial}\cdot\right\rangle_\eta$$

A priori, in this different setting, the crucial value

$$\int_X \mathbb{L}_\eta\left(\chi_{R_0} p_4\right) d\mu_\eta$$

could change, and so the balancing condition would change accordingly. To see that it is not the case we have to compute

$$\int_X \left\langle \mathrm{Ric}_\eta, i\partial\overline{\partial}\chi_{R_0} p_4\right\rangle_\eta d\mu_\eta$$

$$\begin{aligned}
\int_X \left\langle \mathrm{Ric}_\eta, i\partial\overline{\partial}\chi_{R_0} p_4\right\rangle_\eta d\mu_\eta &= \int_X \mathrm{Ric}_{i\bar{j}} g^{i\bar{b}} g^{a\bar{j}} \nabla_a \nabla_{\bar{b}} \chi_{R_0} p_4 \, d\mu_\eta \\
&= \int_X \nabla_a \left[\mathrm{Ric}_{i\bar{j}} g^{i\bar{b}} g^{a\bar{j}} \nabla_{\bar{b}} \chi_{R_0} p_4\right] d\mu_\eta \\
&\quad - \int_X \nabla_a \left[g^{a\bar{j}} \mathrm{Ric}_{i\bar{j}}\right] g^{i\bar{b}} \nabla_{\bar{b}} \chi_{R_0} p_4 \, d\mu_\eta \\
&= \int_X \nabla_a \left[\mathrm{Ric}_i^a \nabla^i \chi_{R_0} p_4\right] d\mu_\eta - \int_X \nabla_a \left[\mathrm{Ric}_i^a\right] g^{i\bar{b}} \nabla_{\bar{b}} \chi_{R_0} p_4 \, d\mu_\eta \\
&= \int_X \partial \left[\mathrm{Ric}_\eta^\sharp \left(\partial^\sharp \chi_{R_0} p_4\right) \lrcorner \, d\mu_\eta\right] - \frac{1}{2}\int_X \nabla_i s_\eta g^{i\bar{b}} \nabla_{\bar{b}} \chi_{R_0} p_4 \, d\mu_\eta \\
&= \lim_{\rho \to +\infty} \int_{\partial X_\rho} \mathrm{Ric}_\eta^\sharp \left(\partial^\sharp \chi_{R_0} p_4\right) \lrcorner \, d\mu_\eta
\end{aligned}$$

We recall that

$$\mathrm{Ric}^{\sharp}_{\eta} = \mathcal{O}\left(|x|^{-4-2m}\right)$$
$$\partial^{\sharp}\chi_{R_0}p_4 = \mathcal{O}\left(|x|^{3}\right)$$
$$d\mu_{\eta} = d\mu_0 + \mathcal{O}\left(|x|^{-1}\right)d\mu_0$$

and so we can conclude that

$$\int_X \left\langle \mathrm{Ric}_{\eta}, i\partial\overline{\partial}\chi_{R_0}p_4\right\rangle_{\eta} d\mu_{\eta} = \lim_{\rho\to+\infty}\int_{\partial X_{\rho}} \mathrm{Ric}^{\sharp}_{\eta}\left(\partial^{\sharp}\chi_{R_0}p_4\right) \lrcorner d\mu_{\eta} = 0$$

We conjecture that we can't have the above case, indeed we conjecture that the decay assumption on η and scalar flatness imply that (X,η) is Ricci-flat.

Conjecture 6.1. Let (X,η) an ALE Kähler space that is scalar flat and such that outside a compact set

$$\omega_{\eta} = i\partial\overline{\partial}\left(\frac{|x|^2}{2} + c|x|^{2-2m} + \mathcal{O}\left(|x|^{2-2m-\gamma}\right)\right) \qquad \gamma>0$$

then (X,η) is Ricci-flat.

If an ALE Kähler space (X,η) is scalar flat, by [BKN89] we have that

$$\omega_{\eta} = i\partial\overline{\partial}\left(\frac{|x|^2}{2} + J|x|^{4-2m} + \mathcal{O}\left(|x|^{4-2m-\gamma}\right)\right) \qquad \gamma>0$$

and if we consider its energy $E(\eta)$ defined (in the real coordinates associated to complex coordinates at infinity) as

$$E(\eta) = \lim_{R\to+\infty}\frac{|\Gamma|}{\mu\left(S^{2m-1}\right)}\int_{\partial X_R}\sum_{i,j=1}^{2m}\left(\partial_i\eta_{ij} - \partial_j\eta_{ii}\right)\nu_j\, d\mu_{\eta}$$

we have

$$E(\eta) = C(m)\,J \qquad C(m)>0$$

So if

$$\omega_{\eta} = i\partial\overline{\partial}\left(\frac{|x|^2}{2} + c|x|^{2-2m} + \mathcal{O}\left(|x|^{2-2m-\gamma}\right)\right) \qquad \gamma>0$$

we have that $E(\eta)=0$. We conjecture that a version of the positive energy theorem could be used to prove that

$$\mathrm{Ric}(\eta) = 0.$$

The proof of the positive energy "a la Schoen" it is not applicable in this setting, but there is hope to adapt the proof that Witten gave in [Wit81]. Witten, in [Wit81], exploits a relation between the Spin structure of a Spin manifold and the energy of the manifold itself, in our case we want to follow this order of ideas and find a relation between the $\mathrm{Spin}^{\mathbb{C}}$-structure on X, its energy $E(\eta)$ and its holonomy. We need better knowledge of scalar flat ALE Kähler spaces and compact cscK orbifolds to understand if the following conjecture is true.

Conjecture 6.2. Let (M,g) be a cscK orbifold with isolated singularities. Let

$$\mathbf{p} := \{p\in M | p \text{ is a } SU(m) \text{ singularity admitting a Kähler crepant resolution }\}$$
$$\mathbf{q} := \{q\in M | q \text{ is a } U(m) \text{ singularity admitting an ALE Kähler scalar flat resolution }\}$$

the set $\mathbf{q}$ can contain smooth points. If

- $\sharp\mathbf{p} + \sharp\mathbf{q} \geq \dim_{\mathbb{C}} \mathfrak{h}_0 + 1$,
- the matrix
$$\Xi := \left(\Delta\Phi\left(\mathbf{p}\right) \quad | \quad \Phi\left(\mathbf{q}\right)\right)$$
has full rank,
- there exist a positive vector $\mathbf{b} \in \ker \Xi$,

then there exist $\left(\tilde{M}, \tilde{g}\right)$ cscK manifold together with a holomorphic, surjective
$$\pi : \tilde{M} \to M.$$
The manifold $\tilde{M}$ is obtained replacing $\mathbf{p}, \mathbf{q}$ with scalar flat ALE-Kähler spaces.

We believe, indeed, that there is a more general and unifying set of conditions relating singular points and potentials of holomorphic vector fields that ensure the existence of a cscK resolution. We think that such a statement can be proved with methods similar to those explained in this thesis, but there are several technical issues. When we try to use techniques of Chapter 3 to construct families of metrics we see immediately that parameters r_ε,R_ε have to be different with respect to the type of point. This fact makes the analysis quite hard and at the moment we can't say if the theory of Chapter 3 can be adapted or new techniques are needed.

Another natural question that arise in the context of desingularization of cscK orbifolds with cyclic singularities is if there is a link with the notion of K-stability for orbifolds that Ross and Thomas introduce in their work [RT11]. The first attempt we can make to understand if this conjectural link is true and in which form is true, in light of the works [Szé12] and [Szé13], is the computing the expansion of the Futaki invariant for the crepant resolutions of cscK orbifolds. We conjecture that, once we have informations on the Futaki invariant, it is possible to perform an analysis in the spirit of [Sto10],[Sto09], [Szé12] and [Szé13] that might give us an answer at least partial.

Appendix A

Elliptic regularity

In this appendix we recall and prove some well known results on elliptic regularity. Most of the material can be found in [BJS79] and [Hör03].

A.1 Fourier Analysis on Tori

Form now on $\mathbb{T}^n$ will be the n-torus $(\mathbb{R}/\mathbb{Z})^n$.

Definition A.1. A formal trigonometric series is an expression of type

$$\alpha(x) = \sum_{\mathbf{l}\in\mathbb{Z}^n} \alpha_{\mathbf{l}} e^{i2\pi\langle \mathbf{l},x\rangle} \qquad \alpha_{\mathbf{l}} \in \mathbb{C} \text{ and } \alpha_{-\mathbf{l}} = \overline{\alpha_{\mathbf{l}}}$$

and we indicate with $\mathcal{S}(\mathbb{T}^n)$ the $\mathbb{R}$-vector space of trigonometrc series. A trigonometric polynomial is a trigonometric series with finitely many $\alpha_{\mathbf{l}} \neq 0$ and we indicate with $\mathcal{P}(\mathbb{T}^n)$ the $\mathbb{R}$-vector space of trigonometrc polynomials.

We define for $t \in \mathbb{Z}$ the t-scalar product between formal trigonometric series α, β

$$\langle \alpha, \beta \rangle_t = \sum_{\mathbf{l}\in\mathbb{Z}^n} \left(1 + |\mathbf{l}|^2\right)^t \alpha_{\mathbf{l}} \beta_{-\mathbf{l}}$$

and the t-norm

$$\|\alpha\|_t = \langle \alpha, \alpha \rangle_t = \sum_{\mathbf{l}\in\mathbb{Z}^n} \left(1 + |\mathbf{l}|^2\right)^t |\alpha_{\mathbf{l}}|^2 .$$

Definition A.2. For $t \in \mathbb{Z}$ we define the spaces $H_t(\mathbb{T}^n) \subset \mathcal{S}(\mathbb{T}^n)$ as completion w.r.t. $\|\cdot\|_t$ of trigonometric polynomials

$$H_t(\mathbb{T}^n) := \overline{\mathcal{P}(\mathbb{T}^n)}.$$

Moreover we define

$$H_\infty(\mathbb{T}^n) := \bigcap_{t\in\mathbb{Z}} H_t(\mathbb{T}^n) ,$$

$$H_{-\infty}(\mathbb{T}^n) := \bigcup_{t\in\mathbb{Z}} H_t(\mathbb{T}^n) .$$

Lemma A.1. *Spaces $H_t(\mathbb{T}^n)$ have the following properties*

- *$H_t(\mathbb{T}^n) \subset H_s(\mathbb{T}^n)$ for $s < t$ and for α, β holds*
$$\|\alpha\|_s < \|\beta\|_t \qquad s < t$$
$$|\langle \alpha, \beta \rangle_t| \leq \|\alpha\|_{t+s} \|\beta\|_{t-s}$$
whenever the left hand side is defined.
- *$H_\infty(\mathbb{T}^n) = C^\infty(\mathbb{T}^n)$.*
- *$H_{-\infty}(\mathbb{T}^n) = \mathcal{D}'(\mathbb{T}^n)$.*
- *For $t \in \mathbb{N}$ we have*
$$H_t(\mathbb{T}^n) = W^{t,2}(\mathbb{T}^n) \,.$$
- *If L is a differential operator on $\mathbb{T}^n$ of order m and $\alpha \in H_t(\mathbb{T}^n)$, then $L\alpha \in H_{t-m}(\mathbb{T}^n)$ and*
$$\|L\alpha\|_{t-m} \leq C(L)\|\alpha\|_t \,.$$

From now on T will be a linear uniformly elliptic operator of order $2m$ on $\mathbb{T}^n$ that is

$$T = \sum_{|I|=2m} a_I(x)\partial^{2m}_{i_1\cdots i_{2m}} + \sum_{\substack{|J|=k\\0\leq k<2m}} a_J(x)\partial^k_{i_1\cdots i_k}$$

such that exists $\theta > 0$

$$\sum_{|I|=2m} a_I(x)\xi^i \cdots \xi^{i_{2m}} \geq \theta|\xi|^{2m} \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}\,, \quad \forall x \in \mathbb{T}^n \,.$$

Theorem A.1. *Let T be a linear uniformly elliptic differential operator on $\mathbb{T}^n$ of order $2m$.*

- *(Garding inequality) There exist $c_1(T), c_2(T) > 0$ constants depending only on T such that for $u \in H_m(\mathbb{T}^n)$*
$$|\langle u, T(u)\rangle_0| \geq c_1(T)\|u\|_m^2 - c_2(T)\|u\|_0^2 \,.$$
- *Let $t \in \mathbb{Z}$, there exist $c_3(T,t) > 0$ constant depending on T and t and $\Lambda(T)$ constant depending only on T such that for $u \in H_t(\mathbb{T}^n)$*
$$\|u\|_t \leq c_2(T)\|T(u) + \lambda u\|_{t-m} \qquad \forall \lambda > \Lambda(T)\,.$$
- *For $\in \mathbb{Z}$ and $\lambda > 0$ sufficienty large the operator*
$$T + \lambda I : H_t(\mathbb{T}^n) \longrightarrow H_{t-2m}(\mathbb{T}^n)$$
is a bounded invertible linear operator with inverse
$$(T + \lambda I)^{-1} : H_{t-2m}(\mathbb{T}^n) \longrightarrow H_t(\mathbb{T}^n)$$
bounded indipendently from λ.

Proposition A.1. *Let $u \in \mathcal{D}'(\mathbb{T}^n)$ and $f \in H_t(\mathbb{T}^n)$ and let $T(u) = f$ then $u \in H_{t+2m}(\mathbb{T}^n)$.*

Proof. By Lemma A.1 exist $k \in \mathbb{Z}$ such that $u \in H_k(\mathbb{T}^n)$. We have that $f + \lambda u \in H_{\min\{k,t\}}(\mathbb{T}^n)$ and if $\lambda > 0$ is sufficiently large by Theorem A.1

$$u = (T + \lambda I)^{-1}(f + \lambda u) \in H_{\min\{k+2m,t+2m\}}(\mathbb{T}^n)$$

and we can repeat the argument until we obtain $u \in H_{t+2m}(\mathbb{T}^n)$. □

A.2 Elliptic regularity in euclidean spaces

Now let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary and T a linear uniformly elliptic operator of order $2m$ on Ω that is

$$T = \sum_{|I|=2m} a_I(x)\partial^{2m}_{i_1\cdots i_{2m}} + \sum_{\substack{|J|=k \\ 0\leq k<2m}} a_J(x)\partial^{k}_{i_1\cdots i_k}$$

such that exists $\theta > 0$

$$\sum_{|I|=2m} a_I(x)\xi^{i}\cdots\xi^{i_{2m}} \geq \theta\,|\xi|^{2m} \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\}\,, \quad \forall x \in \Omega$$

We start with a representation theorem for distributions.

Theorem A.2. *Let $U \in \mathcal{D}\left(\Omega\right)$, then*

- *there exist $t \in \mathbb{N}$ and a constant $C > 0$ depending only on u s. t. for every $v \in C^\infty_0\left(\Omega\right)$*

$$|\langle u, v\rangle| \leq C\,\|v\|_{W^{t,2}(\Omega)}\ ;$$

- *there exist $t \in \mathbb{N}$ and a $u \in W^{t,2}_0\left(\Omega\right)$ s. t. for every $v \in C^\infty_0\left(\Omega\right)$*

$$|\langle u, v\rangle| = \int_\Omega u\Delta^t v\,.$$

Theorem A.3 (Weyl principle)**.** *Let $u \in \mathcal{D}'\left(\Omega\right)$ and $f \in W^{k,2}_{loc}\left(\Omega\right)$ such that*

$$T\left(u\right) = f$$

then $u \in W^{k+2m,2}_{loc}\left(\Omega\right)$. In particular, if

$$T\left(u\right) = 0$$

then $u \in C^\infty_{loc}\left(\Omega\right)$.

Proof. First we assume that $u \in L^2_{loc}\left(\Omega\right)$ so for every $C^\infty_0\left(\Omega\right)$ it is true that

$$\int_\Omega uT^*(v) = \int_\Omega fv\,.$$

Since differentiability is a local property it is sufficient to prove that, in each $\Omega_3 \Subset \Omega$ such that $u \in L^2\left(\Omega_3\right)$ and $f \in W^{k,2}\left(\Omega_3\right)$ and for all

$$\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \Omega$$

then $u \in W^{k+2m,2}\left(\Omega_1\right)$. Let χ_2 be a smooth cutoff s.t. $\chi_2 = 1$ in Ω_2, $\chi_2 = 0$ in $\Omega_3 \setminus \Omega_2$ and $\mathrm{supp}(\nabla(\chi_2)) \subset \Omega_3 \setminus \overline{\Omega_2}$. We have that

$$\int_\Omega f\chi_2 v = \int_\Omega uT^*(\chi_2 v) = \int_\Omega \chi_2 uT^*(v) + uT^*_1(v)$$

with T_1^* a differential operator of order $2m-1$ with coefficients supported in $\Omega_3 \setminus \overline{\Omega_2}$. Since all inegrands have support in $\overline{\Omega_3}$ then for Q a cube such that $\Omega \Subset Q$ and every $w \in C^\infty(Q)$ periodic then

$$\int_Q f\chi_2 w = \int_Q \chi_2 u T^*(w) + uT_1^*(w)\,.$$

We want to use the theory developed for the tori, so we modify slightly T, u, f in such a way they become an elliptic operator and functions on a torus $\mathbb{T}^n$. Now let ρ smooth cutoff function with $\rho = 1$ in Ω_2 and $\operatorname{supp}(\nabla\rho) \subset \Omega_3 \setminus \operatorname{supp}(\chi_2)$. We define

$$\begin{aligned}\hat{u} &= \rho u\\ \hat{f} &= \rho u\\ \hat{T}^* &= \rho T^* + \theta\,(1-\rho)\,\Delta^m\end{aligned}$$

and so now $u \in H_0(\mathbb{T}^n)$, $f \in H_k(\mathbb{T}^n)$ and $\hat{T}$ an uniformly elliptic operator on $\mathbb{T}^n$. We have by construction

$$\begin{aligned}\int_Q \hat{u}\,[\rho T^* + \theta\,(1-\rho)\,\Delta^m]\,(\chi_2 w) &= \int_Q \hat{u}\chi_2\,[\rho T^* + \theta\,(1-\rho)\,\Delta^m]\,(w) + \int_Q \hat{u}\rho T_1^*\,(w)\\ &\quad + \theta\int_Q \hat{u}\,(1-\rho)\,T_2^*\,(w)\\ &= \int_Q \chi_2 u T^*(w) + uT_1^*(w)\\ &= \int_Q \hat{f}\chi_2 w\end{aligned}$$

with T_2^* a differential operator of order $2m-1$ and coefficients supported in $\operatorname{supp}(\nabla\chi_2)$. So we have the following distributional equation on $\mathbb{T}^n$

$$[\rho T + \theta\,(1-\rho)\,\Delta^m]\,(\chi_2\hat{u}) = \chi_2\hat{f} - \hat{T}_3\,(\chi_2 u) - [\rho T_1^* + (1-\rho)\,T_2^*]^*\,(\hat{u})$$

with $\hat{T}_3$ a a differential operator of order $2m-1$ and coefficients supported in $\operatorname{supp}(\nabla\rho)$ and $[\rho T_1^* + (1-\rho)\,T_2^*]^*$ the formal adjoint of the operator $\rho T_1^* + (1-\rho)\,T_2^*$. Since $\hat{u} \in H_0(\mathbb{T}^n)$, then

$$\begin{aligned}[\rho T + \theta\,(1-\rho)\,\Delta^m]\,(\chi_2\hat{u}) &\in H_{-2m}(\mathbb{T}^n)\\ \chi_2\hat{f} - \hat{T}_3\,(\chi_2 u) - [\rho T_1^* + (1-\rho)\,T_2^*]^*\,(\hat{u}) &\in H_{1-2m}(\mathbb{T}^n)\end{aligned}$$

but by Theorem A.1 we have

$$\chi_2\hat{u} \in H_1(\mathbb{T}^n)$$

and so we obtain $u \in W^{1,2}(\Omega_2)$. Now we can iterate this argument choosing open sets contained in $\Omega_2 \setminus \overline{\Omega_1}$ to obtain $u \in W^{k+2m,2}(\Omega_1)$. We come now to the general case in which u is a distribution, then by Theorem A.2 there exist a $\upsilon \in W_0^{t,2}(\Omega)$ for some $t \in \mathbb{N}$ such that

$$\Delta^t \upsilon = u\,.$$

We consider now the operator

$$\tilde{T} = T \circ \Delta^t$$

and we apply the above argument to $\tilde{T}$ to reach the conclusion. □

Theorem A.4. *A linear homogeneous elliptic operator with constant coefficients*

$$L : C_0^\infty\left(\mathbb{R}^n\right) \to C_0^\infty\left(\mathbb{R}^n\right)$$

has inverse

$$S : C_0^\infty\left(\mathbb{R}^n\right) \to C_b^\infty\left(\mathbb{R}^n\right) .$$

Proof. The operator L is of the form

$$L = \sum_{|I|=2k} L_I \partial^I$$

and we indicate

$$p\left(L,\xi\right) = \sum_{|I|=2k} L_I \xi^I .$$

To contruct S we'll construct the Green function G_L for L. We seek for G_L in the following way: we construct a "primitive" $I : \mathbb{R}^n \to \mathbb{R}^n$ for the Green function G_L such that

$$LI = \phi\left(|x|\right)$$

and there exist $k \in \mathbb{N}$ such that

$$\Delta^k \phi\left(|x|\right) = \frac{1}{|x|^{2-n}}$$

so G_L will be

$$G_L = C\left(k,\nu\right) \Delta^{k+1} I .$$

We'll distinguish two cases: n odd and n even.

1. Case $n = 2\nu + 1$
   We define

   $$I(x) := \int_{|\xi|=1} \frac{\left|\langle x,\xi\rangle\right|^{2k+1}}{p\left(L,\xi\right)} d\mu_{S^{2\nu}}\left(\xi\right)$$

   and by definition the function I is clearly C^{2k} in x, it is positively homogeneous of degree $2k+1$ and so by Euler's theorem on homogeneous functions

   $$I(x) = P_{2k+1}\left(x\right)$$

   with P_{2k+1} a homogeneous polynomial of degree $2k+1$. We note, moreover, that for $f \in C^2\left([0,+\infty)\right)$

   $$\partial_{x_i}\partial_{x_j} f\left(|\langle x,\xi\rangle|\right) = \xi_i \xi_j f^{(2)}\left(|\langle x,\xi\rangle|\right)$$

   So we have

   $$\begin{aligned} LI(x) =& L\left[\int_{|\xi|=1} \frac{\left|\langle x,\xi\rangle\right|^{2k+1}}{p\left(L,\xi\right)} d\mu_{S^{2\nu}}\left(\xi\right)\right] \\ =& \int_{|\xi|=1} \frac{L\left[\left|\langle x,\xi\rangle\right|^{2k+1}\right]}{p\left(L,\xi\right)} d\mu_{S^{2\nu}}\left(\xi\right) \\ =& (2k+1)! \int_{|\xi|=1} \left|\langle x,\xi\rangle\right| d\mu_{S^{2\nu}}\left(\xi\right) . \end{aligned}$$

The function $LI(x)$ is clearly continuous and moreover it is $SO(n)$-invariant, and this implies

$$LI(x) = c|x| .$$

Since $n = 2\nu + 1$ we have that

$$\Delta^{\nu+1} LI(x) = c\delta_0$$

and since $I = P_{2k+1}$ we can interchange L and $\Delta^{\nu+1}$ so we set

$$G_L = C(\nu, k)\, \Delta^{\nu+1} I .$$

We also note that for a multi-index I with $|I| < 2k$ we have $\partial^I G_L \in L^1_{loc}(\mathbb{R}^n)$ and for $|I| = 2k - 1$ we have

$$\partial^I G_L(\lambda x) = \lambda^{1-n} \partial^I G_L(x) \qquad \lambda > 0 .$$

2. Case $n = 2\nu$.
We set

$$I(x) := \int_{|\xi|=1} \frac{(A + B\log(|\langle x, \xi\rangle|))\,|\langle x, \xi\rangle|^{2k+2}}{p(L, \xi)} d\mu_{S^{2\nu}}(\xi)$$

with $A, B \in \mathbb{R}$ to be suitably chosen. Clearly I is of class C^{2k} in x and with a scaling argument we can conclude that

$$I(x) = P_{2k+2}(x) + Q_{2k+2}(x)\log(|x|)$$

with P_{2k+2}, Q_{2k+2} homogeneous polynomials of degree $2k+2$. If we apply the operator L to I

$$\begin{aligned} LI(x) =& L\left[\int_{|\xi|=1} \frac{(A + B\log(|\langle x, \xi\rangle|))\,|\langle x, \xi\rangle|^{2k+2}}{p(L, \xi)} d\mu_{S^{2\nu}}(\xi)\right] \\ =& \int_{|\xi|=1} \frac{L\left[(A + B\log(|\langle x, \xi\rangle|))\,|\langle x, \xi\rangle|^{2k+2}\right]}{p(L, \xi)} d\mu_{S^{2\nu}}(\xi) \\ =& A\alpha(\nu, k) \int_{|\xi|=1} |\langle x, \xi\rangle|^2 d\mu_{S^{2\nu}}(\xi) + B\beta(\nu, k) \int_{|\xi|=1} \log(|\langle x, \xi\rangle|)\,|\langle x, \xi\rangle|^2 d\mu_{S^{2\nu}}(\xi) . \end{aligned}$$

And in this case too using $SO(n)$-invariance and scaling arguments we canconclude

$$LI(x) = (A\alpha_1(\nu, k) + B\beta_1(\nu, k))\,|x|^2 + B\beta_2(\nu, k)\,|x|^2 \log(|x|)$$

and we choose A, B st

$$LI(x) = C(k, \nu)\,|x|^2 \left[1 - \frac{2\nu}{\nu + 1}\log(|x|)\right]$$

and because of this we have

$$\Delta^{\nu+1} LI(x) = C(k, \nu)\,\delta_0$$

and so we set

$$G_L := C(k, \nu)\, \Delta^{\nu+1} I .$$

We also note that for a multi-index I with $|I| < 2k$ we have $\partial^I G_L \in L^1_{loc}(\mathbb{R}^n)$ and for $|I| = 2k - 1$ we have

$$\partial^I G_L(\lambda x) = \lambda^{1-n} \partial^I G_L(x) \qquad \lambda > 0 .$$

The inverse S now is

$$S(\phi)(x) = \int_{\mathbb{R}^n} G_L(x-y)\,\phi(y)\,d\mu$$

It is easy to show (integration by parts) that S is the inverse of L .

$\square$

Theorem A.5 (Calderon-Zygmund)**.** *Let K be a singular kernel, $\phi \in L^p(\mathbb{R}^n)$ then for $1 < p < \infty$*

$$\|K \star \phi\|_{L^p(\mathbb{R}^n)} \leq C(K,n,p)\,\|\phi\|_{L^p(\mathbb{R}^n)}$$

Proof. [BJS79]. $\square$

Theorem A.6 (Korn-Lichtenstein-Giraud)**.** *Let K be a singular kernel, $\phi \in C^{0,\alpha}(\mathbb{R}^n)$ then*

$$\|K \star \phi\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C(K,n,\alpha)\,\|\phi\|_{C^{0,\alpha}(\mathbb{R}^n)}$$

Proof. [BJS79]. $\square$

Theorem A.7. *Let $1 < a < \infty$, $1 < p < q < \infty$,*

$$\frac{1}{p} + \frac{1}{a} = 1 + \frac{1}{q}$$

and let

$$K_a(y) = \frac{1}{|y|^{\frac{n}{a}}}\,,$$

then

$$\|K_a \star u\|_{L^q(\mathbb{R}^n)} \leq C_{p,q}\,\|u\|_{L^p(\mathbb{R}^n)}$$

for every $u \in L^p(\mathbb{R}^n)$.

Proof. [Hör03]. $\square$

Theorem A.8. *Let $u \in \mathcal{D}'(\Omega)$ wth Ω open set of $\mathbb{R}^n$ and let $|\nabla u| \in L^p_{loc}(\Omega)$ with $1 < p < n$ then $u \in L^q_{loc}(\Omega)$ for*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{n}\,.$$

Proof. [Hör03]. $\square$

Theorem A.9. *Let $u \in \mathcal{D}'(\Omega)$ wth Ω open set of $\mathbb{R}^n$ and let $\partial^I u \in L^p_{loc}(\Omega)$ for every I with $|I| = k$ and $1 < p < \infty$ then for every J with $|J| < k$ $\partial^J u \in L^q_{loc}(\Omega)$ for $q < \infty$*

$$\frac{1}{p} \leq \frac{1}{q} + \frac{k-|J|}{n}\,.$$

Proof. [Hör03]. $\square$

Theorem A.10. *Let $f \in L^p(B_\rho)$ with $1 < p < \infty$ respectively $f \in C^{0,\alpha}(B_\rho)$ and ρ then $S(f) \in W^{2k,p}(B_\rho)$ respectively $S(f) \in C^{2k,\alpha}(B_\rho)$.*

Proof. We note that we can extend f to 0 to all $\mathbb{R}^n$ with the same regularity and by abuse of notation we call it f again. By Theorem A.7 we have that

$$S\left(f\right)\left(x\right)=\int_{\mathbb{R}^n} G_L\left(y\right)f(x-y)d\mu_y<\infty\,.$$

We prove a fundamental identity for $\phi\in C_0^{\infty}\left(\mathbb{R}^n\right)$ and then the result will follow by density. Since

$$u\left(x\right)=\int_{\mathbb{R}^n} G_L\left(x-y\right)\phi\left(y\right)d\mu_y=\left(-1\right)^n\int_{\mathbb{R}^n} G_L\left(z\right)\phi\left(x-z\right)d\mu_z$$

we have $u\in C^{\infty}\left(\mathbb{R}^n\right)$. Suppose $|I|=2k-2$ then

$$\begin{aligned}
\partial_i\partial_j\partial^I u =&\partial_i\partial_j\left[\int_{\mathbb{R}^n}\partial_x^I G_L\left(x-y\right)\phi\left(y\right)d\mu_y\right]\\
=&\partial_i\partial_j\left[\int_{\mathbb{R}^n}\partial_y^I G_L\left(x-y\right)\phi\left(y\right)d\mu_y\right]\\
=&\left(-1\right)^n\partial_i\partial_j\left[\int_{\mathbb{R}^n}\partial_z^I G_L\left(z\right)\phi\left(x-z\right)d\mu_z\right]\\
=&\left(-1\right)^n\partial_i\left[\int_{\mathbb{R}^n}\partial_z^I G_L\left(z\right)\partial_{x_j}\phi\left(x-z\right)d\mu_z\right]\\
=&\left(-1\right)^{n+1}\partial_i\left[\int_{\mathbb{R}^n}\partial_z^I G_L\left(z\right)\partial_{z_j}\phi\left(x-z\right)d\mu_z\right]\\
=&\left(-1\right)^{n+1}\partial_i\left[\int_{\mathbb{R}^n}\partial_z^I G_L\left(z\right)\partial_{z_j}\phi\left(x-z\right)d\mu_z\right]\\
=&\int_{\mathbb{R}^n}\partial_{x_i}\partial_y^I G_L\left(x-y\right)\partial_{y_j}\phi\left(y\right)d\mu_y\\
=&-\int_{\mathbb{R}^n}\partial_{y_i}\partial_y^I G_L\left(x-y\right)\partial_{y_j}\phi\left(y\right)d\mu_y\\
=&-\lim_{\epsilon\to0}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\partial_{y_i}\partial_y^I G_L\left(x-y\right)\partial_{y_j}\phi\left(y\right)d\mu_y\\
=&-\lim_{\epsilon\to0}\int_{|x-y|=\epsilon}\partial_{y_i}\partial_y^I G_L\left(x-y\right)\phi\left(y\right)\frac{\left(x^j-y^j\right)}{|x^j-y^j|}d\mu_{S_x^{n-1}}\\
&+\lim_{\epsilon\to0}\int_{\mathbb{R}^n\setminus B_\epsilon(x)}\partial_{y_j}\partial_{y_i}\partial_y^I G_L\left(x-y\right)\phi\left(y\right)d\mu_y\\
=&c\left(I,i,j\right)\phi\left(x\right)+\int_{\mathbb{R}^n}\partial_{y_j}\partial_{y_i}\partial_y^I G_L\left(x-y\right)\phi\left(y\right)d\mu_y\,.
\end{aligned}$$

The we divide the proof in two cases $f\in L^p$ and $f\in C^{0,\alpha}$.

- **L^p case:**
  Since
  $$\partial_{y_j}\partial_{y_i}\partial_y^I G_L$$
  is a singular kernel, by Theorem A.5, we have that for $|I|=2k$
  $$\left\|\partial^I S\left(f\right)\right\|_{L^p(\mathbb{R}^n)}\leq C\left(p,L,n\right)\left\|f\right\|_{L^p(\mathbb{R}^n)}\,.$$
  So the distribution $S(f)$ has derivatives of order $2k$ in $L^p\left(\mathbb{R}^n\right)$ then applying Theroem A.9 we have that $S\left(f\right)\in W^{2k,p}\left(B_\rho\right)$.

- $C^{0,\alpha}$ **case:**
  Since
  $$\partial_{y_j}\partial_{y_i}\partial_y^I G_L$$
  is a singular kernel, by Theorem A.6, we have that for $|I| = 2k$
  $$\left\|\partial^I S\left(f\right)\right\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C\left(\alpha, L, n\right)\left\|f\right\|_{L^p(\mathbb{R}^n)} .$$
  So the distribution $S(f)$ has derivatves of order $2k$ in $C^{0,\alpha}\left(\mathbb{R}^n\right)$ then applying Theroem A.9 we have that $S\left(f\right) \in C^{2k,\alpha}\left(B_\rho\right)$.

□

A.2.1 Elliptic operators with variable coefficients

Theorem A.11. *Let L be an elliptic operator of order $2k$ with coefficients in $C^{0,\alpha}\left(\Omega\right)$. Let $\Omega_0 \Subset \Omega$. Let $u \in C^{2k,\alpha}\left(\Omega_0\right)$, then for every $\Omega_1 \Subset \Omega_0$*

$$\|u\|_{C^{2k,\alpha}(\Omega_1)} \leq C\left(L, \alpha, \Omega_0, \Omega_1\right)\left[\|Lu\|_{C^{0,\alpha}(\Omega_0)} + \|u\|_{C(\Omega_0)}\right] .$$

Proof. [BJS79]. □

Theorem A.12. *Let L be an elliptic operator of order $2k$ with coefficients in $C^{0,\alpha}\left(\Omega\right)$. Let $\Omega_0 \Subset \Omega$. Let $u \in W^{2k,p}\left(\Omega_0\right)$, then for every $\Omega_1 \Subset \Omega_0$*

$$\|u\|_{W^{2k,p}(\Omega_1)} \leq C\left(L, p, \Omega_0, \Omega_1\right)\left[\|Lu\|_{L^p(\Omega_0)} + \|u\|_{L^p(\Omega_0)}\right] .$$

Proof. [BJS79]. □

Theorem A.13. *Let L be a strongly elliptic linear differential operator of order $2k$ on $\mathbb{R}^n$, let $f \in L^p\left(\mathbb{R}^n\right)$ respectively $f \in C^{0,\alpha}\left(\mathbb{R}^n\right)$ then there exist a sufficiently small $r > 0$ such that on $B_r\left(0\right)$ we can find a solution $u \in W^{2k,p}\left(B_r\right)$, respectively $u \in C^{2k,\alpha}\left(B_r\right)$ such that*

$$Lu = f \qquad \text{on } B_r .$$

Moreover any two solutions u, u' differ on $B_{r-\epsilon}$ by a function $v \in C^\infty\left(B_{r-\epsilon}\right)$.

Proof. [BJS79]. □

A.3 Elliptic regularity on manifolds

Theorem A.14. *Let (M, g) be a smooth compact n-dimensional riemannian manifold without boundary, let L be a linear elliptic operator of order $2k$ with smooth coefficients. Suppose $f \in L^p\left(M\right)$ for $1 < p < \infty$ (respectively $f \in C^{0,\alpha}\left(M\right)$) and $u \in \mathcal{D}'\left(M\right)$ such that*

$$L\left(u\right)\left[\phi\right] = \int_M f\,\phi\,d\mu_g \qquad \forall \phi \in C^\infty\left(M\right)$$

Then $u \in W^{2k,p}\left(M\right)$ (respectively $u \in C^{2k,\alpha}\left(M\right)$) and satisfies the estimate

$$\|u\|_{W^{2k,p}(M)} \leq C\left(M, g, p\right)\left(\|f\|_{L^p(M)} + \|u\|_{L^p(M)}\right)$$

respectively

$$\|u\|_{C^{2k,\alpha}(M)} \leq C\left(M, g, \alpha\right)\left(\|f\|_{C^{0,\alpha}(M)} + \|u\|_{C^{0,\alpha}(M)}\right) .$$

Proof. The strategy will be the following:

step 1) For each point $x \in M$ find a neighborhood U_x on which i can construct an inverse for the operator L on U_x.

step 2) By compactness select in a smart way a finite number of such U_x.

step 3) Glue these local inverses to get an approximate inverse (with estimates) for L on M.

step 4) Modify the above approximate inverse prove the regularity of u.

step 5) Get the desired estimates.

step 1) Since M is compact we have

$$\operatorname{injrad}_g(x) \geq r_{\min} > r_0 > 0 \qquad \forall x \in M\,.$$

Now we fix $x \in M$, in a coordinate chart centered at x the operator L can be written as

$$L = \sum_{|I|=0}^{2k} L_I(x)\partial^I\,.$$

By Theorem A.13 there exist a $0 < r_x < r_0$ st exist $u_x \in W^{2k,p}\left(B_{r_x}(x)\right)$, respectively $u_x \in C^{2k,\alpha}\left(B_{r_x}(x)\right)$

$$L\left(u_x\right) = f \qquad \text{on } B_{r_x}(x)$$

there exist, indeed, a map S_x right inverse to L such that

$$S_x : L^p\left(B_{r_x}(x)\right) \to W^{2k,p}\left(B_{r_x}(x)\right)\,,$$
$$S_x : C^{0,\alpha}\left(B_{r_x}(x)\right) \to C^{2k,\alpha}\left(B_{r_x}(x)\right)\,.$$

step 2) By compactness we extract a finite number of poins $x_j \in M$ s.t. $B_{\frac{2}{3}r_{x_j}}(x_j)$ cover M and if for j_1, j_2

$$B_{\frac{2}{3}r_{x_{j_1}}}(x_{j_1}) \cap B_{\frac{2}{3}r_{x_{j_2}}}(x_{j_2}) \neq \emptyset \Rightarrow B_{\frac{2}{3}r_{x_{j_1}}}(x_{j_1}) \cap B_{\frac{2}{3}r_{x_{j_2}}}(x_{j_2}) \subset B_{\frac{2}{3}r_{x_{j_1}}}(x_{j_1}) \setminus B_{\frac{1}{3}r_{x_{j_1}}}(x_{j_1})$$

$$B_{\frac{2}{3}r_{x_{j_1}}}(x_{j_1}) \cap B_{\frac{2}{3}r_{x_{j_2}}}(x_{j_2}) \neq \emptyset \Rightarrow B_{\frac{2}{3}r_{x_{j_1}}}(x_{j_1}) \cap B_{\frac{2}{3}r_{x_{j_2}}}(x_{j_2}) \subset B_{\frac{2}{3}r_{x_{j_2}}}(x_{j_2}) \setminus B_{\frac{1}{3}r_{x_{j_2}}}(x_{j_2})$$

step 3) Now we want to create a global approximate inverse from the local data. To do this we take smooth cutoff functions

$$\rho_{j,t,\tau}(x) := \begin{cases} 1 & x \in B_{tr_{x_j}}(x_j) \\ 0 & x \in M \setminus B_{\tau r_{x_j}}(x_j) \end{cases}$$

and we define

$$\chi_{j,\frac{1}{3}}(x) := \frac{\rho_{j,\frac{1}{3},\frac{2}{3}}(x)}{\sum_j \rho_{j,\frac{1}{3},\frac{2}{3}}(x)}\,.$$

Moreover we define

$$S(f) := \sum_j \chi_{j,\frac{1}{3}} S_{x_j}\left(\rho_{j,\frac{3}{4},1} f\right)$$

and we have that

$$\begin{aligned}
L \circ S\,(f) =& L\left[\sum_j \chi_{j,\frac{1}{3}} S_{x_j}\left(\rho_{j,\frac{3}{4},1} f\right)\right] \\
=& \sum_j L\left[\chi_{j,\frac{1}{3}} S_{x_j}\left(\rho_{j,\frac{3}{4},1} f\right)\right] \\
=& \sum_j \chi_{j,\frac{1}{3}} L\left[S_{x_j}\left(\rho_{j,\frac{3}{4},1} f\right)\right] + \sum_j T\left[\chi_{j,\frac{1}{3}}, S_{x_j}\left(\rho_{j,\frac{3}{4},1} f\right)\right] \\
=& \sum_j \chi_{j,\frac{1}{3}} \rho_{j,\frac{3}{4},1} f + \sum_j T\left[\chi_{j,\frac{1}{3}}, S_{x_j}\left(\rho_{j,\frac{3}{4},1} f\right)\right] \\
=& f + \sum_j T\left[\chi_{j,\frac{1}{3}}, S_{x_j}\left(\rho_{j,\frac{3}{4},1} f\right)\right]
\end{aligned}$$

with $T_1[,]$ a linear combination of derivatives of order at most $2k-1$ for $S_{x_j}\left(\rho_{j,\frac{3}{4},1} f\right)$ and up to order $2k$ for $\chi_{j,\frac{1}{3}}$. We define

$$K_r\,(f) := T\left[\chi_{j,\frac{1}{3}}, S_{x_j}\left(\rho_{j,\frac{3}{4},1} f\right)\right]$$

and then we have

$$L \circ S = Id + K_r\,.$$

Similarly

$$\begin{aligned}
S \circ L\,(u) =& \sum_j \chi_{j,\frac{1}{3}} S_{x_j}\left(\rho_{j,\frac{3}{4},1} L(u)\right) \\
=& \sum_j \chi_{j,\frac{1}{3}} S_{x_j}\left[L\left(\rho_{j,\frac{3}{4},1} u\right) + T_2\left(\rho_{j,\frac{3}{4},1}, u\right)\right] \\
=& \sum_j \chi_{j,\frac{1}{3}} S_{x_j}\left[L\left(\rho_{j,\frac{3}{4},1} f\right)\right] + \sum_j \chi_{j,\frac{1}{3}} S_{x_j}\left[T_2\left(\rho_{j,\frac{3}{4},1}, u\right)\right] \\
=& \sum_j \chi_{j,\frac{1}{3}} \rho_{j,\frac{3}{4},1} f + \sum_j \chi_{j,\frac{1}{3}}\left[S_{x_j} \circ L\left(\rho_{j,\frac{3}{4},1} f\right) - \rho_{j,\frac{3}{4},1} f\right] + \sum_j \chi_{j,\frac{1}{3}} S_{x_j}\left[T_2\left(\rho_{j,\frac{3}{4},1}, u\right)\right] \\
\\
=& f + \sum_j \chi_{j,\frac{1}{3}}\left[S_{x_j} \circ L\left(\rho_{j,\frac{3}{4},1} f\right) - \rho_{j,\frac{3}{4},1} f\right] + \sum_j \chi_{j,\frac{1}{3}} S_{x_j}\left[T_2\left(\rho_{j,\frac{3}{4},1}, u\right)\right]\,.
\end{aligned}$$

With $T_2\,(,)$ a linear combination of derivatives of order at most $2k-1$ for u and up to order $2k$ for $\rho_{j,\frac{3}{4},1}$. We define

$$K_l\,(u) = \sum_j \chi_{j,\frac{1}{3}}\left[S_{x_j} \circ L\left(\rho_{j,\frac{3}{4},1} f\right) - \rho_{j,\frac{3}{4},1} f\right] + \sum_j \chi_{j,\frac{1}{3}} S_{x_j}\left[T_2\left(\rho_{j,\frac{3}{4},1}, u\right)\right]$$

and so we have

$$S \circ L = Id + K_l \,.$$

The maps K_r, K_l are "regularizing" indeed

$$K_l : L^p(M) \to W^{1,p}(M) \qquad K_l : W^{2k,p}(M) \to W^{2k+1,p}(M)$$

respectively

$$K_r : C^{0,\alpha}(M) \to C^{1,\alpha}(M) \qquad K_r : C^{2k,\alpha}(M) \to C^{2k+1,\alpha}(M) \,.$$

For every $N \in \mathbb{N}$ we define an operator S_N in the following way

$$S_N := \sum_{a=0}^{N} (-1)^a K_l^a \circ S \,.$$

We have by construction

$$\begin{aligned} L \circ S_N &= Id + (-1)^N K_r^{N+1} \,, \\ S_N \circ L &= Id + (-1)^N K_l^{N+1} \,. \end{aligned}$$

step 4) For every $u \in \mathcal{D}'(M)$ we can find an integer $t \in \mathbb{N}$ sufficiently big such that exist $\upsilon \in W^{t,2}(M)$ and $c_u \in \mathbb{R}$ such that

$$u[\phi] = \int_M \upsilon \Delta_g^t \phi d\mu_g + c_u \int_M \phi \, d\mu_g \,.$$

So if $u \in \mathcal{D}'(M)$ satisfies the equation

$$L(u)[\phi] = \int_M f \phi d\mu_g$$

then it satisfies

$$L(u)[\phi] = u[L^*(\phi)] = \int_M \upsilon \Delta_g^t L^*(\phi) \, d\mu_g + c_u \int_M L^*(\phi) \, d\mu_g = \int_M f \, \phi \, d\mu_g \,.$$

We can apply step 2 to the operator $L \circ \Delta_g^t$ and get an approximate inverse $\tilde{S}_N$

$$\begin{aligned} \tilde{S}_N \circ L(u)[\phi] =& u\left[\left(\tilde{S}_N \circ L\right)^* (\phi)\right] \\ =& \int_M \upsilon \Delta_g^t \left(\tilde{S}_N \circ L\right)^* (\phi) \, d\mu_g + c_u \int_M \left(\tilde{S}_N \circ L\right)^* (\phi) \, d\mu_g \\ =& \int_M \upsilon \left(\tilde{S}_N \circ L \circ \Delta_g^t\right)^* (\phi) \, d\mu_g + c_u \int_M \left(\tilde{S}_N \circ L\right)(1) \phi d\mu_g \\ =& \int_M \upsilon \left(Id + (-1)^N \tilde{K}_l^{N+1}\right)^* (\phi) \, d\mu_g + c_u \int_M \left(\tilde{S}_N \circ L\right)(1) \phi d\mu_g \\ =& \int_M \upsilon \phi d\mu_g + (-1)^N \int_M \tilde{K}_l^{N+1}(\upsilon) \, \phi d\mu_g + c_u \int_M \left(\tilde{S}_N \circ L\right)(1) \phi d\mu_g \\ =& \int_M \tilde{S}_N(f) \, \phi d\mu_g \end{aligned}$$

and so we have

$$\upsilon + (-1)^N \tilde{K}_l^{N+1}(\upsilon) + c_u \tilde{S}_N \circ L(1) = \tilde{S}_N(f) \ .$$

Since L has smooth coefficients, for N big enough we can conclude that υ has the same regularity of $\tilde{S}_N(f)$.

step 5) Now we know that $u \in W^{2k,p}(M)$ respectively $u \in C^{2k,\alpha}(M)$ so

$$L(u) = f \ .$$

By Theorems A.12 and A.11 we have that for each $x \in M$ and r_0 as in step 1 we have

$$\|u\|_{W^{2k,p}\left(B_{\frac{r_0}{2}}(x)\right)} \leq C(L,p,r_0,x)\left[\|Lu\|_{L^p\left(B_{r_0}(x)\right)} + \|u\|_{L^p\left(B_{r_0}(x)\right)}\right] ,$$

$$\|u\|_{C^{2k,\alpha}\left(B_{\frac{r_0}{2}}(x)\right)} \leq C(L,\alpha,r_0,x)\left[\|Lu\|_{C^{0,\alpha}\left(B_{r_0}(x)\right)} + \|u\|_{C\left(B_{r_0}(x)\right)}\right] .$$

By compactness we select a finite number of $x_j \in M$ st $B_{\frac{r_0}{2}(x_j)}$ cover M, then let χ_j be a partition of unity subordinated to this covering, then

$$\begin{aligned}
\|u\|_{W^{2k,p}(M)} &\leq \sum_j \|\chi_j u\|_{W^{2k,p}(M)} \\
&\leq \sum_j C(j)\,\|\chi_j\|_{W^{2k,\infty}\left(B_{\frac{r_0}{2}(x_j)}\right)} \|u\|_{W^{2k,p}\left(B_{\frac{r_0}{2}(x_j)}\right)} \\
&\leq \sum_j C(j)\,C(L,p,r_0,x_j)\,\|\chi_j\|_{W^{2k,\infty}\left(B_{\frac{r_0}{2}(x_j)}\right)} \left[\|Lu\|_{L^p\left(B_{r_0}(x_j)\right)} + \|u\|_{L^p\left(B_{r_0}(x_j)\right)}\right] \\
&\leq \left(\sum_j C(j)\,C(L,p,r_0,x_j)\,\|\chi_j\|_{W^{2k,\infty}\left(B_{\frac{r_0}{2}(x_j)}\right)}\right)\left[\|Lu\|_{L^p(M)} + \|u\|_{L^p(M)}\right] .
\end{aligned}$$

The Hölder case is identical, so the theorem is proved.

□

Theorem A.15. *Let (M,g) be a smooth compact n-dimensional riemannian manifold without boundary, let L be a linear elliptic operator of order $2k$ with smooth coefficients and finite dimensional kernel. Suppose $u \in W^{2k,p}(M)$ for $1 < p < \infty$ (respectively $u \in C^{2k,\alpha}(M)$). If u is L^2-orthogonal to* $\ker(L)$ *then*

$$\|u\|_{W^{2k,p}(M)} \leq C(M,g,L,p)\,\|L(u)\|_{L^p(M)}$$

respectively

$$\|u\|_{C^{2k,\alpha}(M)} \leq C(M,g,L,\alpha)\,\|L(u)\|_{C^{0,\alpha}(M)} .$$

Proof. in the following we'll use the notation $(X_l, \|\cdot\|_l)$ for both $W^{l,p}(M)$ and $C^{l,\alpha}(M)$ and we note that $\ker(L)$ is a closed subspace of X_l for any $l \in \mathbb{N}$. We fix a L^2-orthonormal basis of $\ker(L)$

$$\ker(L) = \langle\{\varphi_1, \dots, \varphi_K\}\rangle$$

and this gives a continuous splitting

$$X_l = \ker(L) \oplus Q_l$$

and a continuous projection

$$\pi_{Q_l} : X_l \to Q_l \qquad \pi_{Q_l}(x) = x - \sum_{i=1}^{K} \varphi_i \int_M x \varphi_i d\mu_g \,.$$

The proof is by contradiction. Suppose there is a sequence $\{x_j\}_{j\in\mathbb{N}} \in Q_{2k}$ st $\|x\|_0 = 1$ and

$$\|L(x_j)\|_0 \leq c(j) \qquad c(j) \to 0\,.$$

By Theorem A.14 we have

$$\|x_j\|_{2k} \leq C\left(\|L(x_j)\|_0 + \|x_j\|_0\right)$$

so by by assumptions

$$\|x_j\|_{2k} \leq C(c(j) + 1)$$

and we can conclude that there exist $\kappa > 0$ st

$$\|x_j\|_{2k} \leq \kappa\,.$$

Since we have compact embedding

$$X_{2k} \hookrightarrow X_0$$

we can extract a convergent subsequence that by abuse of notation we call again $\{x_j\}_{j\in\mathbb{N}} \in Q_{2k}$ whose limit is $x \in Q_0$ and $\|x\|_0 = 1$, indeed, using the continuity of π_{Q_0}

$$\pi_{Q_0}(x) - x = \lim_{j\to\infty} \pi_{Q_0}(x_j) - x_j = 0\,.$$

We have for every $\phi \in C^\infty(M)$

$$\begin{aligned}\int_M x L^*(\phi)\, d\mu_g &= \lim_{j\to+\infty} \int_M x_j L^*(\phi)\, d\mu_g \\ &= \lim_{j\to+\infty} \int_M L(x_j)\, \phi d\mu_g\end{aligned}$$

but since $X_0 \subseteq L^q(M)$ continuously for some $1 < q < \infty$ we have, again for every $\phi \in C^\infty(M)$, that

$$\begin{aligned}\left|\int_M x L^*(\phi)\, d\mu_g\right| &\leq \lim_{j\to+\infty} \|L(x_j)\|_{L^q(M)} \|\phi\|_{L^{q'}(M)} \\ &\leq \lim_{j\to+\infty} \bar{C}(j)\, \|\phi\|_{L^{q'}(M)}\end{aligned}$$

with $\bar{C}(j) \to 0$. We conclude that

$$\int_M xL^*(\phi)\,d\mu_g = 0\,,$$

so x is a distributional solution of the equation

$$L(u) = 0\,.$$

By Theorem A.14 we have $x \in \ker(L)$ and so $x \in \ker(L) \cap Q_0$ that implies $x = 0$, but $\|x\|_0 = 1$, contraddiction. $\square$

Appendix B

Matrix Computations

In this appendix we recall some simple matrix identities we needed to prove Proposition 1.7. A reference for this kind of results is [Ser10]. Let $A \in M_m(\mathbb{C})$ be a $m \times m$ complex matrix then its characteristic polynomial $p_A \in \mathbb{C}[t]$ is defined

$$p_A(t) = \det(tI + A) = \sum_{k=0}^{m} t^{m-k} \sigma_k(A)$$

with $\sigma_k \in \mathbb{C}[x_1, \ldots, x_m]$ the k-th symmetric polynomial on the eigenvalues of A. By Newton identities we have the following formulæ

$$\sigma_2(A) = \frac{1}{2}\left(\operatorname{tr}(A)^2 - \operatorname{tr}\left(A^2\right)\right),$$
$$\sigma_3(A) = \frac{1}{6}\left(\operatorname{tr}(A)^3 - 3\operatorname{tr}\left(A^2\right)\operatorname{tr}(A) + 2\operatorname{tr}\left(A^3\right)\right).$$

Now we consider a matrix A of a particular kind:

$$A = I + \alpha W = \delta_{ij} + \alpha \overline{w^i} w^j \qquad w^k \in \mathbb{C}$$

Proposition B.1. *Let $A \in M_m(\mathbb{C})$ such that*

$$A = I + \alpha W = \delta_{ij} + \alpha \overline{w^i} w^j \qquad w^k \in \mathbb{C},$$

then

$$\det(A) = 1 + \alpha |\boldsymbol{w}|^2$$

and if A is invertible

$$A^{-1} = I - \frac{\alpha}{1 + \alpha |\boldsymbol{w}|^2} W.$$

Proof. We want to calculate $\det(A)$

$$\begin{aligned} \det(A) &= \det(I + \alpha W) \\ &= 1 + \sum_{k=1}^{m} \alpha^k \sigma_k(W) \\ &= 1 + \alpha \operatorname{tr}(W) \\ &= 1 + \alpha |\mathbf{w}|^2 \end{aligned}$$

Now we suppose A invertible and we want to calculate A^{-1}.
First we note that

$$W^2 = |\mathbf{w}|^2 W$$

To identify A^{-1} we use formally the Neumann series

$$\begin{aligned}
A^{-1} &= (I + \alpha W)^{-1} \\
&= \sum_{k=0}^{+\infty} (-1)^k \alpha^k W^k \\
&= I + \sum_{k=1}^{+\infty} (-1)^k \alpha^k |\mathbf{w}|^{2(k-1)} W \\
&= I + \left[\sum_{k=1}^{+\infty} (-1)^k \alpha^k |\mathbf{w}|^{2(k-1)} \right] W \\
&= I + \frac{1}{|\mathbf{w}|^2} \left[\sum_{k=1}^{+\infty} (-1)^k \alpha^k |\mathbf{w}|^{2k} \right] W \\
&= I + \frac{1}{|\mathbf{w}|^2} \left[\frac{1}{1 + \alpha |\mathbf{w}|^2} - 1 \right] W \\
&= I - \frac{\alpha}{1 + \alpha |\mathbf{w}|^2} W \,.
\end{aligned}$$

Previous computations are justified if $|\mathbf{w}|^2$ is small enough but by uniqueness of the inverse the result holds whenever A is invertible.

□

Proposition B.2. *Let $B \in M_m(\mathbb{C})$ with*

$$B = \begin{pmatrix} 0 & b \\ \overline{c}^t & 0 \end{pmatrix} \qquad b, c \in \mathbb{C}^{m-1} \,,$$

then

$$\det(I + B) = 1 - \langle c, b \rangle$$

and if $(I + B)$ is invertible

$$(I + B)^{-1} = I - \frac{B}{(1 - \langle c, b \rangle)} + \frac{B^2}{(1 - \langle c, b \rangle)} \,.$$

Proof. First of all we note

$$B = \begin{pmatrix} 0 & b \\ \overline{c}^t & 0 \end{pmatrix} \,,$$

$$B^2 = \begin{pmatrix} b\overline{c}^t & 0 \\ 0 & \langle c, b \rangle \end{pmatrix} \,,$$

$$B^3 = \langle c, b \rangle B \,.$$

We now calculate $\det(I+B)$.

$$\begin{aligned}
\det(I+B) &= 1+\sum_{k=1}^{m}\sigma_k(B)\\
&= 1+\operatorname{tr}(B)+\sigma_2(B)\\
&= 1+\sigma_2(B)\\
&= 1+\frac{1}{2}\left(\operatorname{tr}(B)^2-\operatorname{tr}\left(B^2\right)\right)\\
&= 1-\frac{1}{2}\operatorname{tr}\left(B^2\right)\\
&= 1-\langle c,b\rangle\ .
\end{aligned}$$

We use now, formally, the Neumann series to calculate $(I+B)^{-1}$

$$\begin{aligned}
(I+B)^{-1} &= \sum_{k=0}^{+\infty}(-1)^k B^k\\
&= I+\sum_{k=1}^{+\infty}(-1)^k B^k\\
&= I-\sum_{j=0}^{+\infty}B^{2j+1}+\sum_{j=1}^{+\infty}B^{2j}\\
&= I-\left[\sum_{j=0}^{+\infty}\langle c,b\rangle^j\right]B+\left[\sum_{j=1}^{+\infty}\langle c,b\rangle^{j-1}\right]B^2\\
&= I-\frac{B}{(1-\langle c,b\rangle)}+\frac{B^2}{(1-\langle c,b\rangle)}\ .
\end{aligned}$$

Previous calculations hold if $|c|^2+|b|^2$ is small enough, but by uniqueness of inverse the result holds if $I+B$ is invertible.

□

We need all the results proved so far to get the following proposition.

Proposition B.3. *Let $M\in M_m(\mathbb{C})$*

$$M=\begin{pmatrix}A & b\\ \overline{b}^t & a\end{pmatrix}$$

with $A\in GL(m-1,\mathbb{C})$, $a\in\mathbb{C}^*$, $b\in\mathbb{C}^{m-1}$, then

$$\det(M)=\det(A)\left(a-\overline{b}^tA^{-1}b\right)$$

and if M is invertible

$$M^{-1}=\left[I-\frac{1}{\left(a-\overline{b}^tA^{-1}b\right)}\begin{pmatrix}0 & aA^{-1}b\\ \overline{b}^t & 0\end{pmatrix}+\frac{1}{\left(a-\overline{b}^tA^{-1}b\right)}\begin{pmatrix}A^{-1}b\overline{b}^t & 0\\ 0 & \overline{b}^tA^{-1}b\end{pmatrix}\right]\begin{pmatrix}A^{-1} & 0\\ 0 & \frac{1}{a}\end{pmatrix}\ .$$

Proof. We calculate $\det(M)$. Using propositions above we have

$$\begin{aligned}\det(M) &= \det\left(\begin{pmatrix} A & b \\ \overline{b}^t & a \end{pmatrix}\right) \\ &= a\det(A)\det\left(I + \begin{pmatrix} 0 & A^{-1}b \\ \frac{\overline{b}^t}{a} & 0 \end{pmatrix}\right) \\ &= a\det(A)\left[1 + \sigma_2\left(\begin{pmatrix} 0 & A^{-1}b \\ \frac{\overline{b}^t}{a} & 0 \end{pmatrix}\right)\right] \\ &= \det(A)\left(a - \overline{b}^t A^{-1} b\right).\end{aligned}$$

If M^{-1} is invertible, using propositions above we have

$$\begin{aligned}M^{-1} &= \begin{pmatrix} A & b \\ \overline{b}^t & a \end{pmatrix}^{-1} \\ &= \left[\begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ \overline{b}^t & 0 \end{pmatrix}\right]^{-1} \\ &= \left[I + \begin{pmatrix} 0 & A^{-1}b \\ \frac{\overline{b}^t}{a} & 0 \end{pmatrix}\right]^{-1} \begin{pmatrix} A^{-1} & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \\ &= \left[I - \frac{1}{\left(a - \overline{b}^t A^{-1} b\right)} \begin{pmatrix} 0 & aA^{-1}b \\ \overline{b}^t & 0 \end{pmatrix} + \frac{1}{\left(a - \overline{b}^t A^{-1} b\right)} \begin{pmatrix} A^{-1}b\overline{b}^t & 0 \\ 0 & \overline{b}^t A^{-1} b \end{pmatrix}\right] \begin{pmatrix} A^{-1} & 0 \\ 0 & \frac{1}{a} \end{pmatrix}.\end{aligned}$$

□

Bibliography

[AP06] Claudio Arezzo and Frank Pacard, *Blowing up and desingularizing constant scalar curvature Kähler manifolds*, Acta Math. **196** (2006), no. 2, 179–228. MR 2275832

[AP09] ______, *Blowing up Kähler manifolds with constant scalar curvature. II*, Ann. of Math. (2) **170** (2009), no. 2, 685–738. MR 2552105

[APS11] Claudio Arezzo, Frank Pacard, and Michael Singer, *Extremal metrics on blowups*, Duke Math. J. **157** (2011), no. 1, 1–51. MR 2783927

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The magma algebra system i: The user language*, Journal of Symbolic Computation **24** (1997), no. 34, 235 – 265.

[BGV04] Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004, Corrected reprint of the 1992 original. MR 2273508

[BJS79] Lipman Bers, Fritz John, and Martin Schechter, *Partial differential equations*, American Mathematical Society, Providence, R.I., 1979, With supplements by Lars Gårding and A. N. Milgram, With a preface by A. S. Householder, Reprint of the 1964 original, Lectures in Applied Mathematics, 3A. MR 598466

[BK13] Gavin Brown and Alexander Kasprzyk, *Graded ring database*.

[BKN89] Shigetoshi Bando, Atsushi Kasue, and Hiraku Nakajima, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Invent. Math. **97** (1989), no. 2, 313–349. MR 1001844

[Cal85] Eugenio Calabi, *Extremal Kähler metrics. II*, Differential geometry and complex analysis, Springer, Berlin, 1985, pp. 95–114. MR 780039

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322

[CS04] David M. J. Calderbank and Michael A. Singer, *Einstein metrics and complex singularities*, Invent. Math. **156** (2004), no. 2, 405–443. MR 2052611

[Dem12] Jean-Pierre Demailly, *Complex analytic and differential geometry*, 2012.

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry. MR 1234037

[Hör03] Lars Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003, Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)]. MR 1996773

[Joy00] Dominic D. Joyce, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. MR 1787733

[Kas10] Alexander M. Kasprzyk, *Canonical toric Fano threefolds*, Canad. J. Math. **62** (2010), no. 6, 1293–1309. MR 2760660

[Kro86] Peter Benedict Kronheimer, *Instantons gravitationnels et singularités de Klein*, C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 2, 53–55. MR 851268

[Kro89] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. **29** (1989), no. 3, 665–683. MR 992334

[LS94] C. LeBrun and S. R. Simanca, *Extremal Kähler metrics and complex deformation theory*, Geom. Funct. Anal. **4** (1994), no. 3, 298–336. MR 1274118

[Mor58] Charles B. Morrey, Jr., *On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. I. Analyticity in the interior.*, Amer. J. Math. **80** (1958), 198–218. MR 0106336

[Pac08] Frank Pacard, *Lectures on connected sum constructions in geometry and nonlinear analysis.*

[Roa89] Shi-Shyr Roan, *On the generalization of Kummer surfaces*, J. Differential Geom. **30** (1989), no. 2, 523–537. MR 1010170

[RS09a] Yann Rollin and Michael Singer, *Constant scalar curvature Kähler surfaces and parabolic polystability*, J. Geom. Anal. **19** (2009), no. 1, 107–136. MR 2465299

[RS09b] ———, *Construction of Kähler surfaces with constant scalar curvature*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 5, 979–997. MR 2538497

[RT11] Julius Ross and Richard Thomas, *Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics*, J. Differential Geom. **88** (2011), no. 1, 109–159. MR 2819757

[Ser10] Denis Serre, *Matrices*, second ed., Graduate Texts in Mathematics, vol. 216, Springer, New York, 2010, Theory and applications. MR 2744852

[Sim91] Santiago R. Simanca, *Kähler metrics of constant scalar curvature on bundles over* $\mathbf{C}\mathrm{P}_{n-1}$, Math. Ann. **291** (1991), no. 2, 239–246. MR 1129363

[Sim04] ———, *Canonical metrics on compact almost complex manifolds*, Publicações Matemáticas do IMPA. [IMPA Mathematical Publications], Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2004. MR 2113907

[Sto09] Jacopo Stoppa, *K-stability of constant scalar curvature Kähler manifolds*, Adv. Math. **221** (2009), no. 4, 1397–1408. MR 2518643

[Sto10] ———, *Unstable blowups*, J. Algebraic Geom. **19** (2010), no. 1, 1–17. MR 2551756

[Szé12] Gábor Székelyhidi, *On blowing up extremal Kähler manifolds*, Duke Math. J. **161** (2012), no. 8, 1411–1453. MR 2931272

[Szé13] ______, *Blowing up extremal Kähler manifolds. II*, arXiv:1302.0760v2 (2013).

[Tia00] Gang Tian, *Canonical metrics in Kähler geometry*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2000, Notes taken by Meike Akveld. MR 1787650

[Wit81] Edward Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402. MR 626707

[WZ04] Xu-Jia Wang and Xiaohua Zhu, *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. **188** (2004), no. 1, 87–103. MR 2084775

Ringraziamenti

Voglio innanzitutto ringraziare i miei genitori per avermi sempre sostenuto e dato la possibilità di a arrivare fin qui. Voglio ringraziare Giulia, per aver sempre creduto in me anche quando non lo facevo io stesso, per la sua pazienza e il suo costante supporto, senza di lei sarebbe stato impossibile superare i periodi difficili e arrivare alla fine di questo Ph.D.. Grazie a Mattia, Fabio, Luca e Alessandro per essere stati straordinari compagni di “viaggio” e aver reso indimenticabili questi anni alla SISSA. Spero di non dimenticare nessuno (nel qual caso chiedo umilmente scusa) ringraziando per aver condiviso con me questi anni: Simone “La Spada”, Stefano “La Piana”, Ivan, Flaviana, Irene, Francisca, Francesco “Frank”, Francesco “Bobby”, Ada, Giangiacomo, Tommaso, Cristiano, Giordano, Antonio, Davide, Matteo, Elisa, Dragana, Riccardo, Mauro, Simone, Ugo, Roberta, Giovanni, Germano, John, Günther, Roby Scussa...